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ABSTRACT

Three addresses by internationally renowned mathematics researchers and a fourth paper on the role of research are presented. Each of the addresses focuses on the learning process, but from different points of view. Heinrich Bauersfeld provides an analysis of the miscommunication inherent in many teacher-student interactions. Four deficient areas of research are discussed. Efraim Fischbein directs attention to the role of intuition in learning. Hans Freudenthal traces the growth of number and geometry ideas in one child. The fourth paper, by Richard Lesh, identifies four problem areas needing research. (Author/MK)

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MATHEMATICS EDUCATION REPORT

Some Theoretical Issues in Mathematics Education: Papers from a Research Presession

Richard Lesh, Editor

Walter Secada, Technical Editor

December 1979

ERIC Clearinghouse for Science, Mathematics and Environmental Education
The Ohio State University
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Mathematics Education Reports

Mathematics Education Reports are developed to disseminate information concerning mathematics education documents analyzed at the ERIC Clearinghouse for Science, Mathematics and Environmental Education. These reports fall into three broad categories. Research reviews summarize and analyze recent research in specific areas of mathematics education. Resource guides identify and analyze materials and references for use by mathematics teachers at all levels. Special bibliographies announce the availability of documents and review the literature in selected interest areas of mathematics education. Reports in each of these categories may also be targeted for specific subpopulations of the mathematics education community. Priorities for development of future Mathematics Education Reports are established by the Advisory Board of the Center, in cooperation with the National Council of Teachers of Mathematics, the Special Interest Group for Research in Mathematics Education, the Conference Board of the Mathematical Sciences, and other professional groups in mathematics education. Individual comments on past Reports and suggestions for future Reports are always welcomed by the Associate Director.

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Some Theoretical Issues in Mathematics Education:

Papers from a Research Presession

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INTRODUCTION

At the 1979 Annual Meeting of the National Council of Teachers of Mathematics, the Council's Research Advisory Committee and the Special Interest Group for Research in Mathematics Education (affiliated with the American Educational Research Association) co-sponsored a research presession. Three internationally renowned mathematics education researchers were invited to present addresses:

- Heinrich Bauersfeld, Institut für Didaktik der Mathematik (IDM), University of Bielefeld, Federal Republic of Germany
- Efraim Fischbein, Tel Aviv University, Israel
- Hans Freudenthal, now retired but until recently Director of IOWO (Instituut Ontwikkeling Wiskunde Onderwijs), Utrecht, Netherlands

The three addresses all focus on the learning process, but from very different points of view. Professor Bauersfeld, using examples from American studies, provides a thoughtful analysis of the miscommunication inherent in many teacher-student interactions. Four deficient areas of research in the teaching-learning process, fundamental problems of research and development, and implications for teacher training are discussed.

Professor Fischbein directs attention to the role of intuition in learning, providing numerous illustrations and a rationale for the importance of intuition as students cope with mathematics. The concept of intuitions, some of its characteristics, a classification schema, and the nature of intuitions are described.

Finally, Professor Freudenthal traces the growth of number and geometry ideas in one child. Situations and the child's reactions are presented, with interpretations and additional comments.

A fourth paper, by Richard Lesh, is also included in this document. He expounds on the role of research and the need for cooperation between practitioners and researchers. Four problem areas needing research are identified.

Significant ideas for researchers -- and teachers -- are provided in each document. Careful study of their details should be made by those planning research and by those teaching, for there are points in each paper which need further exploration and development.

We are pleased to make these papers available so that a wider audience can read them and, we hope, gain new perspectives and ideas.

Marilyn N. Suydam
ERIC/SMEAC

SUPPORTING RESEARCH IN MATHEMATICS EDUCATION

Richard Lesh
Northwestern University

In the July 1978 issue of the Journal for Research in Mathematics Education, John Egsgard, then president of the National Council of Teachers of Mathematics, published an editorial titled, "How Can Research in Mathematics Education Become More Effective?" The editorial was brief, offered no positive suggestions to answer its own questions, and concluded with the statement:

> Until the mathematics education research community can come up with results that will affect the classroom teacher...I do not believe that the Council would be justified in providing additional resources for research. (p. 241)

The "Catch 22" irony in the above article would be comical and easily rejected as nonsense--except that its conclusions resulted from negative, naive, and myopic attitudes about educational research which are significant only because they are popular among influential mathematics educators. Nonetheless, it is foolish to conclude that what we don't know won't hurt us. Teachers and other mathematics educators are justifiably critical about the quality of research in their fields. But to criticize the results of past research, or to criticize the way current research is done, is not the same as criticizing, opposing, or arguing not to support efforts aimed at generating knowledge and information pertaining to priority problems in mathematics education. The latter type of criticism can only minimize the chances that mathematics educators will ever find adequate solutions to their most important problems.

What Is Research in Mathematics Education?

The goal of research is to develop a body of useful knowledge related to important issues in mathematics education. The word "research" often conjures up images of data gathering and data analysis, activities that are too narrow to cope with the more important task of knowledge development. Useful knowledge development involves: (a) identifying important problems in mathematics education, (b) formulating agendas of well-defined (and answerable) questions which build upon one another and which contribute to some existing body of knowledge dealing with the underlying problems, (c) identifying answers that are useful in a variety of contexts--weeding out information that is of questionable validity or usefulness, and (d) communicating the results and conclusions in a way that is meaningful to teachers, researchers, and other mathematics educators.

All of the above research functions are worthy of support by professional organizations like NCTM. It is not acceptable for professional organizations to withhold support for knowledge development until: (a) researchers express their results in a form that is meaningful and useful to teachers, or (b) teachers express their problems in a form that is meaningful and accessible to researchers.

The fact that professional organizations have neglected their roles in knowledge development is one of the primary reasons why so many important practitioner problems have been neglected by research. It is also why so much potentially useful information has been neglected by practitioners.

Among the most important shortcomings of mathematics education research are: (a) many of the most important practitioner problems have been neglected, or (b) for problems which have not been ignored, an overwhelming amount of information may be available from a variety of research perspectives, but still very little may be known (because the various areas use different language to express their results, conceive problems in different ways, and, in general, do not articulate well with one another). Information overload is often a more serious barrier to knowledge development than information scarcity. Lack of cumulativeness has been one of the most obvious negative attributes of existing research in mathematics education.

We should attack the above problems--not attack research. Professional organizations should play important roles (a) to clarify problems that are important to practitioners and to describe them so that they are accessible and meaningful to researchers from a variety of fields, and (b) to criticize, select, organize, reconceptualize, and synthesize information from a variety of research areas, and put them in a form that is accessible and useful to practitioners. For example, the major curriculum development projects of the past decade produced a wealth of useful information and materials which today is nearly inaccessible to teachers. One important goal for research ought to be to identify some of the most important barriers to the diffusion and utilization of innovative materials and useful information. Reinventing the wheel has been a major pastime for mathematics educators--and a major waste of time for both teachers and researchers.

Both of the above roles presuppose close working relationships among researchers and practitioners--and an information flow that goes in both directions, not just from researchers to teachers as Egsgard's editorial suggests. Information from practitioners is needed--not only to identify priority problems through needs assessments, status studies, and opinion polls (CBMS/NACOME, 1975; Suydam & Osborne, 1977; Weiss, 1978), but also to shape the direction of research through ethnographic or naturalistic observation studies designed to clarify what teachers and other mathematics educators really do, really care about, or really think. Many people believe that, in mathematics education, the best practice of the best practitioners is still better than the best theories of the best theorists. Therefore, for greatest effectiveness, mathematics education research should distill as much information as possible from this "practitioners' wisdom"--following a pattern set centuries earlier in physics, chemistry, and other well-established areas of scientific inquiry.

Too often the dialogue between researchers and practitioners is imagined as being restricted to one-way researcher-to-practitioner monologues: a professor (who is typically assumed to be a psychologist-type who has little experience in a classroom) lectures teachers about psychological "do's" and "don'ts" based on the results of single isolated studies. This conception is naive for a variety of reasons. First, the practitioner's side of the dialogue includes parents, administrators, textbook writers, legislators, school board members, and other people who participate in the mathematics education enterprise and whose activities and decisions influence classroom instruction. Second, the researcher's side of the dialogue includes mathematicians, representatives from a variety of different areas of scientific inquiry (e.g., psychology, sociology, anthropology, linguistics) as well as many other educational specialists (e.g., measurement/evaluation/testing specialists). Third, the communication system is cyclic, with the information from practitioners to researchers being equally as important as that from researchers to practitioners--especially in the planning or formative stages of knowledge development projects. Fourth, because of the complexity of most of the important problems in mathematics education, it is unrealistic to expect that they be resolved by single isolated studies. In fact, it is unrealistic to expect most individual studies to have immediate and wide-ranging implications for classroom practice. However, it _is_ reasonable to expect individual research studies to feed into a theory or body of knowledge which will have implications within some reasonable length of time (e.g., 5-10 years). If progress is ever to be made on the issues important to mathematics educators, most issues will require long-term intensive commitments and coordinated research efforts from _groups_ of researchers, representing a variety of practical and theoretical perspectives, building upon one another's work over extended periods of time.

For practitioners who are involved in the mathematics education enterprise--whether they are parents, teachers, administrators, legislators, or others--a great deal of information is available which is seldom used. For example, a recent status study (Suydam & Osborne, 1977), funded by the National Science Foundation, concluded that:

1. Educational policy is frequently determined without collecting enough information to allow the process to be rational.

2. Educational policy is frequently constructed without using information that is readily available.

3. Policy formulation typically has ignored existing practices in the schools except as mirrored in the disquietude of society. Information has been collected after-the-fact of policy decision to confirm the actions taken.

Similar statements could be made about other potential practitioner-consumers of research information and materials--including teachers. For example, research or teacher decision making has investigated how teachers' "implicit theories" influence the following kinds of issues: What cues do teachers consider and disregard as they make classroom decisions? Why are some cues more salient to teachers than others? How do teachers weigh and manipulate the cues they consider? What effects do teachers' judgments have on their teaching styles, on classroom behaviors, or on student learning (Morine, 1976; Peterson, Marx & Clark, 1978; Shavelson, 1973; Shavelson, Caldwell & Izu, 1977; Taylor, 1970; Zahorik, 1975)? These studies consistently found that teachers' thoughts during instruction attended primarily to their own behavior, the unpredictable parts of lessons, and needed adjustments in lessons that were going poorly. Teachers made only slight modifications in their plans during instruction, modified only those lessons that were going poorly, and used student involvement cues to determine whether lessons were going poorly. It has also been found that one of the best ways to improve teacher decision-making, in terms of both the quantity and the quality of the information teachers use, is to improve their "implicit theories."

The useful results of research need not always result in a set of pedagogical "do's" and "dont's." In fact, teachers may be the ones best equipped to make these decisions, provided they are given accurate information and useful ways to think about their activities.

Some mathematics educators who claim to be good teachers insist that research has had little influence on their teaching. Such claims usually represent a naive conception of the varieties of products resulting from research. Ever, time a teacher teaches and every time a set of instructional materials is developed, the teacher or authors operate on some basic assumptions (perhaps unarticulated) about teaching and learning. There is little doubt that educators, parents, teachers, government officials, and others throughout the world see reality differently and talk about it differently as a result of the work done by several outstanding researchers. Freud, Dewey, Thorndike, Skinner, Piaget, and Mead are but a few notable examples of individuals whose research has obviously left its mark on both thought and practice in education. However, these influences did not result from isolated studies. Rather, they were the products of theory development which organized, synthesized, and interpreted the results of many studies which were conducted over many years.

Professional organizations have powerful resources which could be used to encourage knowledge development and the formation of research communities to address priority problems in their areas. Through their national and regional meetings, publication outlets, professional reward structures, and a variety of potentially prestigious standing committees and ad hoc committees and projects, professional organizations should contribute to the identification of long-range research agendas to address priority problems. They should help recruit first-rate researchers and practitioners to work together cooperatively on issues of common concern; they should promote the formation of communities of research/practitioners; they should facilitate communication among different types of individuals within these groups; they should help overcome the fragmentation and lack of cumulativeness which has characterized past knowledge-development efforts; and in other ways they should help focus, coordinate, and facilitate the development of useful knowledge dealing with important problems in mathematics education.

Too often, professional organizations have played exactly the opposite kinds of roles. They have alienated "outside" resources; they have defined teacher problems in ways that are short-sighted and naive; and they have helped to popularize superficial and simplistic "solutions." Egsgard's article, "Problems of Teachers of Mathematics and Some Solutions" (1978b), is a convenient example of these latter characteristics Egsgard's "problems" and "solutions" are riddled with references to nice-sounding but naive constructs like "the experienced teacher"--as though teachers could be classified into just two categories, experienced and inexperienced; and as though a teacher who has taught college-bound 11th and 12th graders in Canada is equivalent to a teacher who has taught remedial math to 7th and 8th graders in inner-city Chicago, or to a kindergarten teacher in rural Indiana. Egsgard naively assumes that "good" teachers automatically make good teacher trainers (presumably, by the same line of reasoning, good football players automatically make good coaches). He assumes that a teacher who is good at lecturing to college-bound high school students will be equally good in primary school classes. He also assumes that only one of the following types of individuals and expertise are important in teacher training: (a) people who know a great deal about the mathematics content which is to be taught, (b) people who know a great deal about the psychological capabilities and characteristics of students and/or teachers at a particular grade level, (c) people who have a great deal of experience teaching at a particular grade level or with a particular type of student (e.g., gifted, learning disabled, remedial, etc.), (d) people who know about how the mathematics instruction in a particular class fits into the overall curriculum, (e) people who know about the efficient and effective use of instructional resources (e.g., textbooks, computers, tests, etc.).

No single individual is likely to have all of the above knowledge and experience. Yet, all of these perspectives (and more) should have meaningful input into the solutions of important problems that face mathematics teachers. No single type of individual--teacher, psychologist, mathematician, textbook editor, test developer--should be excluded from the dialogue; rather, professional organizations should create mechanisms to evaluate critically the validity and usefulness of claims made by all these individuals.

It is commendable that some individuals lobby strenuously for the views of particular constituencies within professional organizations, but such efforts should not degenerate into attempts to silence other constituencies or to prejudge the usefulness of their inputs.

The mathematics education enterprise includes a variety of different types of players with different experiences and expertise, and if the system is to function properly, each contributor should focus on those roles that he or she does best. The kinds of skills, abilities, and experiences that are needed to teach a kindergarten class are quite different from those required to write a geometry book based on recent Soviet "teaching studies" research, or from the research skills required to "follow the mathematical thinking" of gifted 7th graders during non-routine problem solving attempts. If teachers and researchers are to fulfill their own roles properly, they cannot be expected also to provide for all of the many other functions that are needed in the mathematics education enterprise.

It is important to build a community of mathematics educators who will work together to generate knowledge and information about important problems in mathematics education. Professional organizations, like NCTM and AERA, have critically important roles to play in the formation of this community.

Problems in Mathematics Education

What are some characteristics of emerging problems in mathematics? The problems that confront mathematics teachers today are similar to those that confront the society at large. They must accommodate an increasingly complex enterprise that has a multiplicity of tasks, ranging from socialization to increasing test scores, despite declining resources. Educators need to reverse values once associated with continuous growth and redirect attention toward finding more efficient and less costly solutions to problems. Unfortunately, the kinds of "solutions" provided by past research has too often required overworked teachers to work harder, and bankrupt school systems to become more expensive.

The world of the late 1970's is quite different from the 1950's and 1960's when massive amounts of money were allocated to the development of new curriculum materials and to the training of more and better teachers. In the late 1950's there were apparent mathematics personnel shortages, school enrollments were increasing rapidly at all levels, and textbooks were badly out of date. Today, many of these trends have been reversed.

During that past decade some of the most powerful influences on mathematics education have been demographic (declining enrollment), economic (dwindling resources), political (equity issues), legal (required special education offerings), and technological (television, calculators, computers). The "baby boom" that flooded our schools in the 1960's has evolved into a middle-aged society in which adult education, continuing education, remedial education, non-school education, and preschool education have become increasingly more important. New student populations have emerged (e.g., adults) who have new educational demands (e.g., mathematics for career opportunities), and new educational institutions have emerged to meet these needs. For example, two-year colleges, community colleges, and a variety of non-certification adult education programs have more than doubled in the past decade. According to the 1976 Databook for the National Institute of Education.

> Education is today the major occupation of 62.2 million people in the United States. That figure, along with the fact that more

> than $96 billion will be spent by educational institutions this year, lends credence to the contention that education is now the Nation's largest enterprise. (p. 5)

Unfortunately, our schools have been slow to adapt to the above "greying of America" phenomena with its new student populations and new educational demands. Consequently, <u>schools</u> are a declining industry at the same time that <u>education</u> is a booming enterprise.

The above trends have important implications for mathematics education research: e.g., mathematics education is not restricted to schools, students are not restricted to children, and instruction is not restricted to teaching. This is not to say that mathematics educators should abandon their traditional concerns with the teaching of youngsters in schools. But, it does mean that mathematics teaching exists within a larger educational system, that it is strongly influenced by non-school factors, and that solutions to problems which do not in some way take these factors into account are likely to be simplistic--and ultimately not helpful.

According to a recent series of NSF-funded needs assessments and status studies (Helgeson, Stake & Weiss, 1978; Stake & Easley, 1978; Suydam & Osborne, 1977; Weiss, 1978), some of the most important difficulties confronting mathematics teachers are related to the following trends:

1. Declining enrollments have resulted in teacher job insecurity and a slowing down of teacher turnover. Few new teachers have been hired; the average age of faculties is increasing, and many teachers opt to teach out of their area in order to maintain employment.

   On the other hand, mathematics teacher shortages have developed in many parts of the country, and talented college students (who might have gone into teaching ten years ago) are no longer getting teaching certification. Once they have lost their teaching jobs and have found other employment, many of the most talented teachers choose not to go back into teaching. So, the reserve pool of talented teachers is often illusory.

   One important role for research is to furnish accurate and meaningful information to describe current circumstances and future trends in education.

2. Pressure for accountability has increased markedly within the past ten years. Therefore, the goals selected for instruction are often the ones that are easiest to document. Standardized tests have assumed increasing importance in spite of the recognition that scores from tests are being misused.

   A second role for research is to construct and validate useful measurement and evaluation instruments.

3. School financial problems have produced larger classes, more courses per teacher, lower salaries, fewer fringe benefits, fewer para-professional assistants, diminished inservice opportunities, and relatively ineffective teacher support systems. School overhead (heat, upkeep, supplies) costs more now than previously. The result is an evaporation of discretionary funds and fewer replacements of texts and equipment. Curriculum development efforts during the past 20 years have produced a number of useful and innovative materials that are today quite inaccessible to most teachers.

   A third role for research is to investigate barriers to the development, dissemination, and utilization of effective and efficient instructional materials.

4. Equal opportunity concerns (for women, minorities, the handicapped) are related to definitions of basic skills. Mathematics has served as a significant barrier for career opportunities. Increased emphasis on equal educational opportunity has also resulted in more heterogeneous classrooms and pressures to treat all students in the same way. Teachers are faced with conflicting pressures to individualize instruction on the one hand, while treating all students alike on the other. In these circumstances, basic skills may be geared to the lowest common denominator of ability levels.

   A fourth role for research is to identify the skills and abilities needed by a variety of different student populations--and to investigate effective ways of meeting these needs.

Many other problem areas could be identified for mathematics education research--ranging from careful descriptions of children's primitive conceptions of particular mathematical ideas (e.g., rational numbers, measurement concepts) to experiments involving teacher-training programs or teacher decision making. However, from the problem areas and trends that have been given already, it is clear that:

1. Perhaps the most general and fundamental challenge now facing mathematics education research is that of achieving better understanding of highly complex phenomena that involve a large number of interacting components (i.e., systems of "organized complexity") in which the extent and severity of problems are often unknown, diffuse, and shifting. Because of interdependencies in the education system, it is increasingly difficult to find solutions to one problem that do not aggrevate or create a new problem. People who should be encouraged to address problems in mathematics education should include more than psychologists or those who call themselves mathematics education researchers--it also should include economists, information dissemination specialists, etc.

Users of mathematics education research information should --if the interests of mathematics teachers are to be served--include not only teachers but also administrators, parents, school board members, legislators, etc.

2. Because of the interdependencies characterizing problems in mathematics education and because of the rapid rates of change, problems need to be anticipated rather than discovered. However, the task of foreseeing problems and predicting policy outcomes is immensely more difficult than the task of reacting to events and adjusting policies by trial and error. Many problems that can be foreseen have so far shown only a small part of themselves. Popular attention and governmental concern tend to focus on these current manifestations of problems--even though they are often little more than precursive symptoms--with the result that actions intended as remedial are often halfway measures. It is this--the response to symptoms--that gives the impression of moving from crisis to crisis, each more unexpected than the last.

Conclusion

Throughout this paper the factors that have been emphasized are those which stress the need for building a community of people who will work together to generate useful knowledge and information about priority problems in mathematics education. Professional organizations have important roles to play in this effort. Yet, these responsibilities have been neglected. It is long past time for positive action.

At the Annual Meeting of the National Council of Teachers of Mathematics, research presessions have been co-sponsored by the NCTM's Research Advisory Committee and the American Educational Research Association Special Interest Group for Research in Mathematics Education. These presessions represent one positive step toward satisfying some of the research support roles described above. The papers that follow were presented at the 1979 presession in Boston by three internationally prominent mathematics education researchers.

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HIDDEN DIMENSIONS IN THE SO-CALLED REALITY OF A MATHEMATICS CLASSROOM

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A Nearly True Story

In old Russia two men meet in a train somewhere between Moscow and Warsaw. Since the beaver collars indicate they are both merchants, one of them asks: "Where are you going?" "To Moscow," the other replies. "Hey," says the first one, "if you say you go to Moscow you must really want me to believe that you go to Warsaw. But this train is headed for Moscow and this makes it certain that you travel to Moscow. So, why are you lying to me?"

These two men are talking not only about directions, but more, they are concerned with their mutual expectations and with their subjective interpretation of what they "really" do. Though the replying man tells the "truth" from our contextual view, the asking protagonist understands the utterance as a lie.

Let us try an explanation. (This is not to kill a joke by explaining, but rather to use the explanation to illustrate a more important problem.) Competing merchants will hardly disclose good sources and addresses to each other. The questioner therefore expects a non-destination as an answer. Knowing these rules of the interaction and hearing the obviously true destination, he must construe a lie. Thus we laugh about a man who seems to be the captive of his expectations. He became accustomed to this game and for him it is reality, his reality. Seen from a more general point of view our "truth" about what is the case is no better or more valid than is his "truth"--although we enjoy a larger majority supporting our interpretation.

This story about "situations," "rules," "expectations and interpretations," and "subjective realities" brings me directly to my theme: hidden dimensions in the so-called reality of a mathematics classroom. After a short overview of mathematics learning as a social activity and the role of related theory, the constitutive power of human interaction will be concretely demonstrated with a documented classroom situation. Following this, four deficient areas of research in mathematics education will be identified and discussed with a view to changing paradigms of research. Finally, I will come back to my main concern of pre-service and in-service teacher training and make some preliminary conclusions for it.

The Contribution of Social Sciences

To view the learning and teaching of mathematics as a social process --a "jointly produced social settlement" as Lee S. Shulman puts it (1979, Note 1)--seems to be a fairly recent issue. Although the ancient Greeks provided us with famous examples of mathematics instruction through dialogue (e.g., Plato's "Meno" in Hamilton and Cairns, 1961), we still do not have much information about the social dimensions of generating mathematical knowledge and of developing individual mathematical power within the classroom. Particularly, researchers in mathematics education have not spent much time studying these dimensions of human interaction. Other disciplines have produced relevant research specific to themselves, although not to the learning of mathematics. Speaking of hidden or neglected dimensions within mathematics education is only relatively true in the sense that researchers have not made use of relevant developments in the other disciplines.

Examples of such contributions come from symbolic interactionism (Blumer, 1969); Goffman, 1969), linguistics (Gumperz & Hymes, 1972; Herrlitz & Gotterts, 1977), ethnomethodology (Cicourel et al., 1974; Mehan & Wood, 1975; Mehan, 1979). The demarcation among these disciplines is difficult, because of their increasing integration through interdisciplinary procedures, procedures which might also benefit mathematics education. Topics such as the generation of meaning and the function of language in social situations, the actual shaping of behavior and cognitive performance through interaction, the specificity of communication in institutionalized settings, etc., apparently force interdisciplinary approaches and have formed a new type of human science.

There is a final point to make in this initial overview. From the very beginning my concern is both pragmatic as well as highly theoretical. It is pragmatic since my goal is to improve mathematics teaching and learning through both teachers' and students' actions. It is theoretical because the "improvement" and the "differentiated orientation" require the most sophisticated, reproducible theoretical framework available. Both aspects, the pragmatic and the theoretical, action and reflection, are deeply interwoven. Albert Einstein has put it sharply: "It is always the theory which decides what can be observed" (Mehra, 1973, p. 269). From a physicist one might expect to hear the complementary statement: It is always the observation (or the "reality") which decides the theory. In the human sciences different actions and different concerns often produce different theories, and different theories in turn produce different realities.

This point is often expressed in education by saying that research findings, like a theory on certain classroom events, need special transformation into teaching practice; or, "that there is little direct connection between research and educational practice" (Kerlinger, 1977, p. 5); or, "...what is good theory for one purpose is not a good theory for another" (Hilgard, 1976, Note 2). All of these statements are only different expressions in an educational setting of the general point made by Einstein.

Bauersfeld

The Constitutive Power of Human Interaction

Two dissertations mark cornerstones for the discussion of human interaction in the mathematics classroom: George Bernard Shirk's "Examination of conceptual framework of beginning mathematics teachers" (1972), and Stanley Erlwanger's "Case studies of children's conception of mathematics" (1974). Both were directed by Jack A. Easley, University of Illinois.

Erlwanger's case studies are related to programs for Individual Prescribed Instruction. His documentation of students' mathematical misconceptions and deficiencies demonstrates how mathematics learning can be damaged by restricted teacher-student communication, a restriction which leads to the near-total absence of negotiations over meanings. It should be clear that the fading fascination shown for programmed instruction fails to provide a satisfactory explanation for the nonappearance of further research with such case studies.

Shirk's work with beginning teachers gives a striking example of the influence of subjective theories about mathematics teaching, the student's role, and the teacher's role. Moreover, his documents give a feel for the fragility of classroom discourse and of the impact of these social situations on mathematics learning. Therefore, I will take a brief example from Shirk's transcripts and use it for comments based on theories from other human sciences.

The episode presents an early part of a beginning teacher's lesson with eighth graders at Urbana Junior High School. The topic is about slides, flips, and turns from "Motion Geometry" (a product of Max Beberman's UICSM), written by Russel Zwoyer and Jo McKeeby Phillips. preceding lesson Tom, the teacher, has defined parallel lines in terms of slides. The lesson under discussion opens with students working on positive examples. The episode which I am going to analyze starts with line 39 of the transcript. The teacher presents a counter-example, two intersecting lines (see Figure 1).

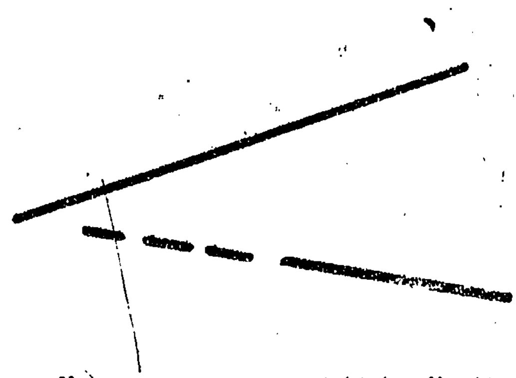

Figure 1.--"Tom, tape of 4/4/72" (from G. B. Shirk, 1972, pp. 173-174)

T - teacher, Tom K - student, Kevin R - student, Reggie

T: ... ... look at the next figure, right below it. Ya, ... now, are those two lines parallel?

K: Nope.

T: Why not?

K: They cross each other.

T: OK, but, uh, ... according to what I've said about parallel lines, what can't they do?

K: Can't cross them ... (?)

T: What?

K: Can't .. they won't, they won't .. I'd rather not .. (ID)

T: What Reggie?

R: Um.

K: ... They won't come together ..

T: OK, but what did I say, what did I say on the first figure? What could you do to get from one line to the other?

K: Slide.

T: OK. There's a slide arrow that'll go, .. that'll take one line into the other.

K: ... Arrow.

T: Right? Is there a slide arrow, ... on the second figure? ... Reggie? On the second figure, can you draw a slide arrow that'll go from one of these lines to the other?

R: (?) Not any more.

T: Like the ... a slide arrow, ... will that take the, ... will that go from one to the other?

R: I don't know.

T: Well, you remember what a slide arrow did?

R: Hm?

T: It, ... it moved a figure along that slide. Can you draw one that'll do that?

R: Oh.

T: OK, is that a slide arrow?

R: (?)

T: What is that? What, ... what did you draw?

R: A circle.

T: Well, you remember.

R: (Laughter)

76 T: ... What that was called?

77 R: (?)

78 T: Does anybody remember ..

79 R: A rotating ...

80 T: OK, we called it a what? ... A turn?

81 R: A turn arrow.

82 T: OK, so what you were starting to draw in was a turn arrow,
83 right? But I'm just talking about slides. Can you draw a
84 slide arrow? Just a straight line, a straight arrow,
85 that'll go from one, that'll take one line to the other?

86 (pause for students to work)

First, I shall follow Shirk's own interpretation and then add my comments later:

> In this episode, Tom was working for a compound goal which he wanted the students to reach: ... that they recognize the lines illustrated were not parallel and realize there was a reason for it through the definition of slides. The students did accomplish the first part of this goal; but when they invoked a reason other than that which Tom sought, it was rejected by him as being inappropriate. The students weren't connecting the lessons together. Tom could only see this as a completely unexpected deficiency in the students' understanding of a lesson which he believed had been learned earlier, so he turned the lesson toward this deficiency in an effort to correct for it. (Shirk, 1972, p. 46)

> The critical aspect centers around Tom's expectation that the students would know all of the consequences of the definition of "parallel." Not seeing this, Tom interpreted their problem as having to do with slides; and this bothered him for he believed that slides had been adequately covered and, therefore, the students should know them. He was also assuming that the students would appreciate everything that he said and therefore, the problem would have to lie elsewhere, i.e., in their more basic preparations which he thought had been covered earlier. (Shirk, 1972, p. 46)

For the re-analysis it is useful to note the major shifts in the student-teacher interpretations of the situation. The episode then splits into four parts.

Part I, lines 39-51: The teacher does not succeed in using the counter-example to infer that intersecting lines cannot be parallel (there is no slide arrow which would move the lines together). Unexpectedly, he receives a much simpler answer, not invoking motion geometry concepts, "they cross each other" (line 43). Albeit correct, the teacher rejects the answer as inadequate "on the basis only that they should remember what he said in the previous example" (Shirk, 1972, p. 47).

The students become confused and uncertain as evidenced by K's stammering and brief withdrawal (line 48).

Part II, lines 52-64: "Tom is now directing the students' attention toward the drawing again in an effort to get them to see the connection between it and the slide" (Shirk, 1972, p. 48). Repeatedly he uses the key word "slide arrow."

The students try to guess the teacher's intentions. Their answers are short and cautious: "Slide" (line 54) and "..arrow" (line 57). Their uncertainty increases. Thus under the teacher's pressing Reggie modifies his answers from "Not any more" (line 61) to "I don't know" (line 64). "With their initial efforts rejected, and Tom emphasizing slides, the students begin to look around for ways to slide the two lines together for there was nothing in the earlier portions of the lesson about 'no slide' or 'not parallel' " (Shirk, 1972, p. 48).

Part III, lines 65-86: Still, the teacher has not given up his initial aim. His impulse, "Well, you remember" (line 74) is an attempt "to get Reggie to put together the formal principles by pointing out to him that what he had drawn were turns rather than slides" (Shirk, 1972, p. 49). Reggie's "failure" (as seen from the teacher's eyes) justifies the causal ascription that the students have forgotten all about slides. Therefore, the teacher begins to "reteach" the concept towards the end of this section.

The students, however, "in an effort to come up with the answer they thought Tom was looking for (namely, a slide arrow), invented slide lines between the two intersecting lines" (Shirk, 1972, p. 49), as did Reggie at line 69 (see Figure 2). Reggie's misinterpreting the teacher's question (lines 74-76) "What that was called?" is completely in line with his looking for slides. The drawings of the students in this part (see Figure 3) expose the extent to which the teacher's pressing has contributed to "spoiling" the students' concept of "slide" and of "slide arrow." Nevertheless, the teacher's interpretation that the students have not learned their slide lesson is not diminished but rather reinforced.

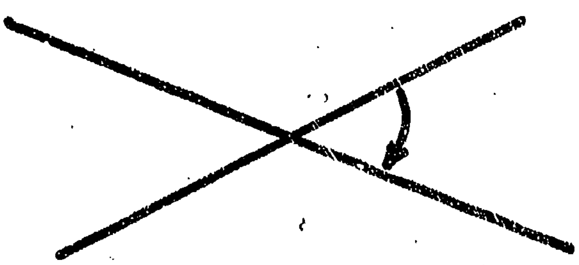

Figure 2. "Reggie at line 69"

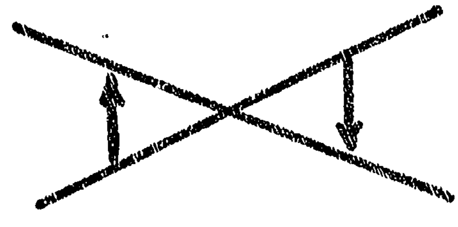

Figure 3. "Students at line 85"

Bauersfeld

Part IV, lines 87-155: (This concluding portion of the episode is not included in the above quote.) "The remainder of this first episode involves Tom's attempts to reteach the concept of slides to the students" (Shirk, 1972, p. 51). And h works on that rehearsal until he gets the conviction that "he has reconnected the students with slides and parallel lines" (Shirk, 1972, p. 52).

On a more general level Shirk explains the episode using the terms "split personality" and "guessing ahead" (the latter from John Holt, 1964).

> A "split personality" ... occurs when the teacher is teaching one lesson and the students, in an effort in "psych out" the teacher, are actually learning another ... The "split personality" in the lesson occured as a result of the conceptual frameworks which governed Tom's actions, rather than being a part of those frameworks themselves. Tom, acting in accordance with his frameworks, interpreted the students' behavior in a certain way and then acted in a manner consistent with his conceptual frameworks. For their part, the students attempted to guess ahead and therefore acted differently. These actions resulted in one path being taken by the students while Tom was trying to lead them along another. (Shirk, 1972, p. 43)

In the above analysis Shirk uses the classical relation of cause and effect as he matches causes with personal attributes. That is, he traces the outcomes back to properties and actions of single individuals. As a result he is led to somewhat discouraging conclusions and recommendations. From his finding that "there was no change discernible within the conceptual frameworks", he concludes that the future teacher education programs "must be so designed so as to be assimilable to the preexisting conceptual frameworks" (Shirk, 1972, p. 165). I shall try an alternate answer to the teacher-training problem later, but first let me give an interpretation of the episode from a different paradigm of social action.

A description of the situation as constituted through the interaction of the participants can challenge the usual causal model, cast doubt on predictive conclusions, and possibly shed light on the use of language in the mathematics classroom.

<u>The constitution of the social situation</u>. Principally, and taken as a piece of an ongoing process, the episode cannot be reconstructed sufficiently--neither from personal variables, from characteristics of the single participant, or from the documented speech production. Hence, from additional interviews with the teacher, and from essays and "comment cards" which the teacher had to write, Shirk has distilled the teacher's conceptualization of mathematics education, of his role as a teacher, and of the student's role. Shirk uses this set of statements only to explain the teacher's moves.

We do not have any comparable information about the students. However, the interactive nature of the process and the mutual relatedness of expectations and interpretations can be partially reconstructed from the transcript. For the purpose of this analysis consider the table on the following page.

The first two lines in Table 1 reconstruct the teacher's interpretation and the students' interpretation of the four parts. (A more detailed analysis can be developed through following the discussion step-by-step.) Comparing the mutual interpretations in columns gives a rough but sufficiently clear idea.

The teacher's immediate objectives change following his changing interpretations of the process. "Guessing ahead" the students' interpretation of the teacher's intentions changes as well. By no means are the actions of the two sides, teacher and students, reactions only to the preceding move of the other side. It is commonly believed that individuals react to the actions of another when in fact they react to their self-constructed interpretation. Yet, "reaction" is misleading. Far from the simple model of stimulus-response, the participant's actions in this social situation are generated through complicated, internal reflective activity. This subjective reflective activity takes into account not only the actual and perceptable moves of the others but also the more general interpretations of the situation, and one's own role in that situation. Furthermore, actual interpretation of related former experiences exercise an influence on the current ongoing interpretation. Each participant's actions contribute to the change of the other's, of their interpretation and their actions. And through this process they contribute to the change of the participant's own interpretation and action. Thus it becomes reasonable to speak of the "constitution" of the social situation (Mehan & Wood, 1975). More precisely: The social situation is constituted at every moment through the interaction of reflective subjects. Ethnomethodologists therefore describe "reality as a reflexive activity" (Mehan & Wood, 1975, p. 8).

The episode under discussion is an example of the constitutive power of human interaction, i.e., the interaction constructs the subjects' various realities. Both teacher and students act according to their subjective realities. The students draw turn arrows for slides; the teacher diagnoses learning deficiencies; and the teacher and students work clearly at cross-purposes convinced they understand the situation clearly.

> Every moment is mysterious, as the understood horizon of the moment is inexhaustible. Every interpretive act indexes this mystery in an unpredictable way. A person's every action is thus creative; it reflexively alters the world. The person begins with certain materials that set limits, and then acts and in acting alters those limits. (Mehan & Wood, 1975, p. 203)

> Forms of life are always forms of life forming. Realities are always realities becoming (Melvin Pollner in Mehan & Wood, 1975, p. 32).

Table 1

Interpretations and Changes

Aspects of analysis	I: lines 39-51	II: lines 52-64	III: lines 65-86	IV: lines 87-155
Teacher's Interpretation	Use counter-example to strengthen concept. Disappointment about student's failure.	Prompting will help the students catch on. Increasing disappointment.	They don't really know what a slide is. Confusion.	Give up and reteach the concept of "slide." Resignation.
Student's Interpretation	Having not treated counter-examples they think about easier descriptions.	There must be something to say or to do with "slide arrows."	He insists on arrows which connect the lines.	He wants us to play the "recitation game."
The actual task, teacher's view	Connect the non-existence of a slide to the non-parallelism of the lines.	Draw the student's attention to the role of the slide arrow.	Find out that there is no slide arrow for intersecting lines.	Help the students reconstruct and recorrect the concept of "slide."
The actual task, student's view	Tell if intersecting lines are parallel	Find another description. Later confusion, "Don't know."	Look for new arrows which might match intersecting lines.	Tell what you have learned previously--"like we said it yesterday..."
Meaning of "slide arrow" for student	Move in positive examples one line onto the other. 	(Opening and changing.)	Any move, straight or turn or up-and-down.	(Like in part I? but scarred and more divergent.)

The constitution of meaning. Not only subjective interpretation and assessment change during the process but also the aims, the actual tasks, and even the concepts. A comparison of the teacher's and the students' views during the episode of the task (see Table 1, third and fourth lines) emphasizes the semantic change of the problem situation. Clearly, each participant's view of the actual task to be done is different and they vary during the course of the episode. The task must be understood as a function of the situation.

For the students, the concept of "slide arrow" also varies across the episode (see Table 1 fifth line). In the beginning the previous experiences with parallel lines (and slide arrows moving them together) is dominant. The intervention of the counter-example and the following discussion with the teacher produce doubts about where to locate the borderlines of the concept. What is and what is not to be included? The students' interpretation of the teacher's insistent questions increasingly spoils the concept and leads to an arbitrary guess as to its true meaning. Any type of arrow, curved or up-and-down, is used by the students (see Figures 2 and 3).

Without further information about the students' thinking, the effect of the reteaching is difficult to evaluate. Surely the residual status of the concept "slide arrow" will differ from its initial status, but it might not be improved. Due to the high affective load during part II, the concept now might be vulnerable to future misunderstanding in similar situations.

Thus, the logical principle of identity is not applicable. The word "slide arrow" does not mean the same to every participant. Moreover, the meaning changes during the episode repeatedly and remarkably. But, if problems and concepts become functions of the situation instead of being constant and stable, it then becomes necessary to consider the social constitution of meaning, i.e., the constitution of meaning through human interaction.

Herbert Blumer (1969) makes the same point: "Symbolic interactionism sees meanings as social products, as creations that are formed in and through the defining activities of people as they interact" (p. 5). However, as a matter of principle, there is small chance of predicting the outcomes of such episodes at their beginning. Nor is there much chance of making predictions about a later stage from the basis of a preceding one. Since we cannot ascribe the constitution of meaning to one single participant (e.g., the teacher), we are not in a position to use causal models as adequate descriptors of individual social interaction, particularly not of mathematics teaching and learning. "As every day meanings do not meet the canons of logic, they are transformed by literal description. These transformed meanings are amenable to causal models. Every day life is not" (Lehman & Wood, 1975, p. 66).

At this point the analysis leads into a revolution of the fundamental paradigm (following Thomas S. Kuhn, 1962). In the human sciences rules are different than in the natural sciences. For example, they are rules about the constituting of situations and meanings rather than rules about the situations and meanings themselves. They are rules about structuring the process rather than about the structure of the process (see Mehan, 1978). In the human sciences the interpretation-assessment paradigm will replace the cause-effect paradigm borrowed from the natural sciences.

<u>The role of language</u>. The above episode also prompts a new look at the role of language in mathematics teaching. Years ago linguistic research would have used the utterances in Shirk's episode for a syntactical and a semantic analysis of the material. For example, the analysis might have pointed to a lack of adjectives. Or, it might have noted that the units of speech consist only of short or broken sentences, that paratactical structures dominate, or that many deictic words appear (words which demonstrate or point to something--"here," "this," "there," etc.). Without any additional information the analyst would speak of a "restricted code" and perhaps a resulting poorly developed meaning.

It is the fundamental idea in Chomsky's theory (1957) that we cannot reconstruct the constitutional process of communication from the surface, that is, from recorded speech. Neither "discovery-models" (for the discovery of new grammars) nor "decision-models" (for the decision about the adequacy of a grammar) will work. Chomsky maintains that these models only make use of the linguistic data. Since we have to analyze the rules of structuring communication, we must analyze evaluative structures of individual interpretation and assessment, and this analysis requires an "evaluation-model" (for the evaluation among existing grammars). At this point there is a change of paradigm from objective (linguistic) data to interpretative structures as the object of analysis. Since Chomsky's work, sociolinguists have increasingly studied the use of language in social interaction.

An interesting and helpful issue is the concept of <u>indexicality</u>. If there is no further information, then most people will not understand the discourse in lines 41-51 of the episode:

41 K: Nope.

42 T: Why not?

43 K: They cross each other.

44 T: OK, but, uh, ... according to what I've said about parallel
45 lines, what can't they do?

46 K: Can't cross them ... (?)

47 T: What?

48 K: Can't ... they won't, they won't ... I'd rather not ... (?)

49 T: What Reggie?

50 R: Um.

51 K: ... They won't come together ... (Shirk, 1972, p. 173)

Mathematics educators may even become doubtful about the topic and meaning of the discussion if they don't know that the document is related to mathematics instruction. Bar-Hillel (quoted in Mehan & Wood, 1975) defined such "utterances that require contextual information to be understood" as "indexical expressions." Thus an "informed" outsider can even have difficulty understanding what is going on in a discussion. It is difficult to realize what the participants intend to say and to identify the meaning they create in the given situation. As a prerequisite for communication, participants have to share common understanding which they take as an implicit basis of reference when speaking to each other. While speaking, each participant anticipates the understanding and the interests of the specific addressee. The speech gets organized through the expectation of what the addressed person already knows. Each speaker uses his interpretation of the given situation and of the addressee as an index from which he forms his utterances and from which he decides his "choice of grammar."

In the classroom, the "teacher's instructions are indexical expressions which requires teachers and children to employ contextually bound interpretative practices to make sense of these instructions" (Cicourel et al., 1974, p. 129). What a participant says not only transports the intended message, but over and above the message the utterance contains information about his understanding of the topic, his interpretation of the situation, his expectations of what the others might know, as well as his present emotional concerns.

Hence, indexicality is another label for the thesis that <u>the situation influences how language is used</u>. Not only are content and meanings negotiated and constituted in the social situation, but also the use of language and the performance of the speaker are co-determined. This is true for both the syntactical structure of the utterances and for the actual choice of words. (From this point of view Bernstein's distinction (1973, 1965) between "elaborated" and "restricted" codes might be more an issue of the indexical and reflexive constitution of the situation rather than of the competence of the speakers.

Mathematicians, in particular, have invested much effort in producing universal statements, and most school mathematicians would claim any mathematical statement as non-indexical, i.e., as universal and objective. However, this conviction blocks insight into the irreparable incompleteness of utterances, and more general, of any symbolic action. Each utterance, just as each symbolic form, is necessarily incomplete, because it has to be filled in with meaning via contextual interpretation. Through its genesis and chain of definition a concept inevitably gets infiltrated with contextual information. And "every attempt at repair increases the number of symbols that need to be repaired" (Mehan & Wood, 1975, p. 93). Therefore, understanding mathematics is not only a case of logic, or of divergent thinking, or of proper definitions. As far as understanding is realized in social interaction (or through communication, which is the very same thing) it inescapably becomes dependent upon the interpretative, indexical, and reflexive constitution of meaning.

Four Deficient Areas of Research--Mathematics Education's Hidden Dimensions

The process of teaching and learning mathematics can be viewed most aptly as a highly complex human interaction in an institutionalized setting--an interaction which forms a distinctive part of the participant's life. Four aspects in this issue deserve more detailed discussion since they represent weak areas of research.

1. Teaching and learning mathematics is realized through human interaction. It is a kind of mutual influencing, an interdependence of the actions of both teacher and student on many levels. It is not a unilateral sender-receiver relation. Inevitably the student's initial meeting with mathematics is mediated through parents, playmates, teachers. The student's reconstruction of mathematical meaning is a construction via social negotiation about what is meant and about which performance of meaning gets the teacher's (or the peer's) sanction. "Symbolic interactionism sees meaning as social products, as creations that are formed in and through the defining activities of people as they interact" (Blumer, 1969, p. 5). How can we expect to find adequate information about teaching and learning when we neglect the interactive constitution of individual meanings?

2. Teaching and learning mathematics is realized in institutions which the society has set up explicitly to produce shared meanings among their members. Institutions are represented and reproduced through their members and that is why they have characteristic impact on human interactions within that which is institutional. Institutions establish norms and roles; they develop rituals in actions and in meanings; they tend to seclusion and self-sufficiency; and they even produce their own content, in this case, school mathematics. How reliable are studies on the effects of mathematics education if they do not take into account the institutional impact on teacher and student? The question becomes crucial when one thinks about any application of knowledge learned at school to situations outside the school.

3. Mathematics education constitutes a distinctive part of the student's life as well as the teacher's. Anyone who is active in mathematics will learn something about himself, especially since the activity happens in interactive situations. On the other hand one can learn mathematics only by actively engaging his previous knowledge of related subjects and actions. Therefore, mathematics education is deeply related to the man-made world of symbols and meanings, to common sense, and to everyday life. Mathematics education depends on our social and historic conditions. How can we dare to make any prediction about the mathematical abilities of a student and

about his chances to develop these abilities if not by carefully relating such statements to his personality and background?

4. Scientists are not the only ones who have difficulties dealing with highly complex issues. The orientation for actions and decisions in the classroom continuously requires the reduction of complexity. On the other hand, the understanding and the effective reduction of complexities demands their total unfolding and complete reconstruction. To date scientific analysis has been incapable of reducing the complexities of an actual mathematics classroom sufficiently for guiding a teacher's decisions. Yet without such guidance it is impossible to plan effective teacher-training programs.

Fundamental Problems of Research and Development

Within the last years, my view of the structure of the classroom process has changed as a result of collecting information from several disciplines, participating in mathematics lessons, and analyzing video-taped mathematics lessons. This subjective change includes the aims of my work, the subject to be studied, the methods of research, as well as the underlying paradigms of my thinking. This personal event is worthy of mention since discussions with colleagues leads me to believe that my subjective difficulties only mirror much more fundamental difficulties within our profession. Philosophers of science agree that such difficulties within a profession are strong indicators of fundamental change of paradigms.

In the present transition stage three theses seem to be of importance:

1. Mathematics education is deeply in need of theoretical orientation. We have too much research on too small a theoretical basis. Many opening addresses of APA, AERA, and SIG/RME, as well as journal articles from within the last years have complained about this problem, e.g., Lee Cronbach (1975), Lee S. Shulman (1979, Note 1), Ernest Hilgard (1976, Note 2). Perhaps there are too many short-termed research contracts. Perhaps there is too much prescription for "acceptable" research programs. Or, perhaps, there is no support for methodological heretics and thus no encouragement for young researchers to try unusual approaches. For sure, there has yet to appear an adquate forum for theoretical discussion (compared with West German publications, the United States provides for very little discussion of metatheory.) Whatever the cause, I do not believe that there are not enough new ideas.

2. Research and practical developments follow different paradigms. The main stream of research still follows the paradigm of the natural sciences, stating an objective educational reality, using well-defined and quantifiable concepts, and analyzing the relationship among them through statistical means. For a long time we have heard and accepted complaints about the complete lack of classroom applicability of research results. (See Kerlinger, 1977, for a sum-up from a researcher's view.)

On the contrary, the majority of a teacher's classroom decisions are made via common sense and intuition rather than through rational analyses by scientific means. If she/he is a good teacher then her/his actions are based on a more differentiated perception of the classroom events than research recognizes. She/he is more open to contextual changes, "knowing" a student, using "tacit knowledge" (M. Polanyi, 1966) and informal reflected experiences. Compared with these "hard" social facts, current research appears as "soft-ware." It is necessary that research in mathematics education takes notice of this gap if a claim for practical relevance is to be established.

3. Interdisciplinary approaches are promising, if not necessary, for closing mathematics education's "credibility gap." Within the broad area of social sciences, the discrepancies between rationalistic and hermeneutic descriptions, between naive and scientific constructs, have been realized and investigated much earlier. It is time to integrate these findings into our profession and to transform this knowledge to the specific conditions of learning and teaching mathematics.

Unlike the natural sciences, the human sciences must deal with an objective social reality on the one hand, yet on the other hand must deal with as many realities as there are reflective subjects. Paradoxically, modern physicists have a highly developed understanding of explanatory models losing their meaning in the light of more comprehensive theories. For example, the question of the "divisibility" or the "consistency" of a light quantum (photon) makes no sense in a general theory of elementary particles, because the theory describes the relations among elements but is not concerned with the nature of the elements themselves. This is very near to an important issue of constitutive ethnomethodology. The structuring activities of the participants form ("constitute") the social situation among themselves, and the process and rules of these "structurings" (as Mehan calls them) build a core theory of social action. The theory is related to structurings rather than to structures of the situation in usual social sciences. That is, there is greater generalizability within the process of structurings than within the structures themselves.

Implications for Teacher Training

Those who find the discussed theories and interpretations more or less acceptable might find themselves forced to think about consequences. "It seems likely, that innovation in schools will not be of a very radical kind unless the categories teachers use to organize what they know about pupils and to determine what counts as knowledge undergo a fundamental change" (Keddie, 1971, p. 156). Shirk's (1972) findings about the stability of fundamental conceptualizations across teacher training apparently do not leave much chance for that change. How is it possible for a beginning teacher to overcome the sixteen or more years of his own experiences as a student? This problem is especially difficult, since the contextual force of these experiences often dominates any later verbal information about education and leads the beginner into an almost unconscious reproduction of the school system's characteristics.

But, if we form our cognition and behavior about teaching through social situations, then we can also change this formed cognition and behavior through social situations. We learn to behave in social settings only through the reflected participation and action in social settings. Similarly, a teacher will learn to teach or to change his teaching pattern only through reflected teaching. Yet, this is not the ruling model of present pre-service teacher training.

Usually the student teacher learns about teaching in contexts very different from classroom situations. The organizing interest for picking up knowledge in lectures is more along with passing examinations than related to later classroom application. Through various lectures and seminars the student teacher collects incomplete eclectic knowledge and she/he is left with the unassisted task of integrating this knowledge into an applicable system for the living classroom.

If the constitutive power of social situations on behavior, meaning, and language is as strong as assumed here, then the student teacher will have to spend much more time planning, accomplishing, and reflecting upon real classroom teaching experience. From the very beginning the teacher-to-be must encounter an adequate complexity of social classroom exchanges. "Adequate" means that the complexity of the teaching-learning situation might be reduced in quantity, e.g., via a reduced number of students to teach or a reduced amount of lesson time, but not reduced in quality (as simulation games or video-tape analyses, e.g., would cause). If we claim to educate human beings, then a teacher will have to receive a much more careful, holistic preparation.

This, of course, will require support and development on the side of research as well. And this research, at least a reasonable part of it, will have to follow the interpretive paradigm. "Science at its best is thus like a firm but gentle hand that holds a butterfly without crushing it" (Kenneth S. Bowers, 1973, p. 332).

Reference Notes

1. Shulman, Lee S. Presentation at the SIG/RME pre-session of the NCTM annual meeting, Boston, April 18, 1979.

2. Hilgard, Ernest R. Presentation at the Symposium on 75th Yearbook of NSSE, AERA annual meeting, April, 1976.

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INTUITION AND MATHEMATICAL EDUCATION*

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The Concept of Intuition

Though the concept of intuition has various meanings and has been defined in various forms, there is a common feature which is always mentioned: intuition is immediate knowledge. In other words, an intuitively accepted truth is self-evident; its acceptance does not require any explicit proof. For instance, we accept directly the statement: "The shortest way between two points is the straight line," or "It is always possible to find a natural number greater than any given natural number."

A great part of mathematical axioms are based on such self-evident, intuitively accepted truths. On the other hand, there are various mathematical truths which contradict intuition, and learners often have difficulty accepting these truths. For instance, it is difficult to accept that the set of natural numbers is equivalent to the set of positive even numbers. It is difficult to accept that the set of points of a segment is equivalent to the set of points of a square or of a cube.

My opinion is that the intuitive reaction of the learner cannot be neglected by mathematical education for the following reasons:

a) Wrong intuitions render difficult the acquisition of correct interpretations in a given field. Even if the student has succeeded in learning the scientifically correct version, it is not certain that a primitive, false interpretation has been eliminated. The survival of such wrong interpretations may endanger the adequate use of correct knowledge, especially in non-standard situations.

b) Correct intuitive interpretations are able to stimulate productive mathematical thinking. Pure, formal, symbolic representations of mathematical truths are, by themselves, not efficient as mental tools, especially when the solution to non-standard problems is requested.

Some Characteristics of Intuition

Some general properties characterizing intuition are:

1) Self-evidence. This basic characteristic was already mentioned.

2) Coercive effect. As a consequence of their intrinsic obviousness, intuitions exert a coercive effect on the processes of conjecturing, explaining, and interpreting various facts.

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3) Extrapolative capacity. Intuition, like analytical thinking transcends direct, empirically obtained information. Specific to intuition, however, is that it represents a mental leap which cannot be completely justified by logical or factual arguments.

4) Globality. Intuition is described as a global, synthetic view, as opposed to analytical thinking. Being a condensed view, intuition is frequently expressed by visual symbolization.

5) High stability. Intuitions often exhibit high resistance to teaching influences and to formal experimentations. As already has been mentioned, primitive interpretations may remain active even after the student has acquired corresponding, correct information. We shall return to that point later.

A Classification of Intuitions

A. Several categories of intuitions can be identified.

a) Affirmatory intuitions are self-evident representations, interpretations of explanations. The previously cited examples refer to affirmatory intuitions. Let us add some more examples. It appears as being self-evident that: "The opposed angles formed by two intersected lines are equal." Or, "In a triangle, each side is smaller than the sum of the two other sides." Finally, "Through a point outside a line, one, and only one line can be drawn, which is parallel to the original."

b) Anticipatory intuitions are preliminary, global views which precede the analytical, fully developed solution of a problem.

c) Conclusive intuitions summarize, in a globally structured vision, the basic ideas of the solution to a problem previously elaborated.

B. A second classification refers to the origin of affirmatory intuitions.

a) Primary intuitions are interpretative or explanatory beliefs which naturally develop in human beings, before and independent of systematic instruction. Such intuitions are profoundly influenced by the cultural setting. For instance, we have a natural, trimensional non-isotropic representation of space. Another primary intuitive representation, weight, is an intrinsic quality of objects. This does not mean that primary intuitions are fixed forever. On the contrary, they are largely dependent on our personal experience. What we mean when speaking about primary intuitions is that such intuitions develop naturally in the child as a result of basic daily life experiences.

b) Secondary intuitions are those which are developed by systematic intellectual training (generally, in the school setting). For instance, to a physicist it seems natural to affirm that a body keeps moving with constant direction and velocity if no force intervenes. This is a secondary intuition, which contradicts the primary intuition that a body keeps moving only if a force acts on this body.

The above classification implies the following fundamental hypothesis: Intuitions--though appearing as givens (i.e., as produced by some a priori intellectual mechanisms) are, in fact, changeable. They may be built, transformed, corrected, or eliminated as a result of adequate training.

A corollary is that even learned truths, elaborated by a highly complex conceptual system, can attain the characteristics of intuitions, i.e., become accepted as natural self-evident truths.

C. Operational and content-oriented intuitions.

a) Operational intuitions are those which express the feeling of validity which accompanies logical operations. For instance, in a syllogism the conclusion is determined by the premises, but the validity of the syllogism, as a method of deducing a truth from previously accepted premises, cannot be proved; we must accept it by intuition (Ewing, 1941, in Westcott, 1968, pp. 17-19). It is by intuition that we accept the universality of inductive inferences.

Generally speaking, the axioms of logical thinking are based on such fundamental beliefs. Mathematical education should not be satisfied with training blind, automatic intellectual skills corresponding to the laws of logical thinking. New intellectually adequate beliefs, i.e., intuitions, must be built in correspondence to the learned truth tables. For instance, it is easy to teach the pupil the truth table of implication. But the problem is that he must get accustomed to feel the rules of implication as obvious and to act accordingly. Such intuitions must be developed as content-free mental structures.

b) Content-oriented intuitions refer to representations, explanations, or interpretations which express--in a correct or in a wrong manner--our mental attitudes toward reality. Elementary space intuitions, elementary chance evaluations, explanations and interpretations of physical phenomena accepted as self-evident, etc. belong to this category. While operational intuitions are related to the formal schemes of our logical inferences, content-oriented intuitions are related to phenomena as such. Of course, in concrete cognitive processes both categories are strongly interrelated. There are intuitively meaningful mathematical operations which belong simultaneously to both of these categories. For example, many geometrical transformations can be completely integrated in a logical reasoning without losing their iconic significance. The statement, "The product of two line reflections is a translation", has a full pictorial intuitive meaning. At the same time, it is a formal, a priori acceptable truth.

The Nature of Intuitions

Do intuitions really have essential features in common, or has language simply equated various forms of cognitive reactions on the basis of some external, accidentally similar aspects? Intuitions represent the basic mental mechanisms for connecting knowledge with action. Strictly speaking, knowledge is image, i.e., an internal, subjective replication of objective realities. Concurrently, knowledge is basically oriented towards action: the basic role of knowledge is to help prepare for action. In the case of sensory perceptions, the transition from representation to action may be direct: the perceptive (sensory) representations of the concrete features of real objects and events describe--and thus prepare--our possible actions on them (on the basis of previous experience). By discriminating and synthesizing signals, by evaluating distances and intervals, we are preparing and eliciting adapted, efficient reactions. Generally, perceptions are so directly implicated into action (by their origin and structure) that by themselves, they can prepare and direct action. In Pavlov's terminology, perceptions constitute the first signal system.

It is not the same with symbolic forms of knowledge and, particularly with logical, analytical thinking. Solving a problem by analytical procedures is, more or less, a lasting process. An explicit, logical process is time-consuming. Very often it cannot be effective if a direct, prompt, rapid form of adaptation is required. In my opinion, the essential role of intuition is to translate cognitive acquisitions into terms of action. Intuitions share essential features with the iconic forms of knowledge, enabling them to translate information directly in terms of practical decisions. The following example illustrates this point.

Suppose that I intend to cross the street, and I look left to see if any vehicles are approaching. I quickly evaluate the number of vehicles, their (decreasing) distance, their speed, and the width of the street I intend to cross. In less than a second I get something which may be termed as a "behavioral conclusion." I decide to cross the street or I decide to continue to wait. If I decide to move, the decision includes speed and direction. In fact, in such a situation, perception, evaluation, decision-making, and effective behavior are deeply interconnected. When estimating the distance and the speed of the approaching cars, the distance I have to cover in order to reach the opposite sidewalk, etc., I get a unique, global representation of the whole situation. This global evaluation includes an anticipation of the efforts and reactions which will enable me to cope correctly with the situation. The perceived, estimated distance is, in fact, a direct estimation of my own efforts to cover that distance, a direct preparation for appropriate, effective reactions.

Perception does not need any intervening connective for guiding action. Perception is by itself the beginning of action. The iconic and the enactive modes of representation are deeply interconnected by sensorimotor mechanisms. It is not the same with the symbolic form of representation. By its very nature, symbolic representation belongs to a different system of connections and means of expression than the iconic and the enactive. An intervening mechanism of translation between symbolic and enactive is needed, and this is intuition.

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Intuition is, simultaneously, a derived form of cognition--as thinking is--and a programme for action, as perception is. Intuition and perception have essential common features and for this reason the term intuitive knowledge is sometimes used for denominating both categories. Both are global, direct, effective forms of information.

The differences between intuition--as a specific form of knowledge--and perception is that intuition does not directly reflect an object or an event with all its concrete qualities. Intuition is mostly a form of interpretation, a solution to a problem, i.e., a derived form of knowledge like symbolic knowledge. On the other hand, the difference between intuition and analytical thinking is that intuition is not analytical, not discursive, but rather a compact form of knowledge like perception. Like perception, intuition does not require extrinsic justification. With perception we have the feeling of being plunged directly into the world of material objects. Perception appears to be reality itself rather than just appearance. With intuition, we have the same feeling of being in the object and not a simple interpreter of it.

Being a derived form of knowledge like analytical thinking, intuition can organize information, synthesize previously acquired experiences, select efficient attitudes, generalize verified reactions, and guess, by extrapolation, beyond the facts at hand. The greatest part of the whole process is unconscious and the product is a crystallized form of knowledge which, like perception, appears to be self-evident, internally structured, and ready to guide action.

In its anticipative form, intuition offers a global perspective of a possible way of solving a problem and thus, inspires and directs the steps of seeking and building the solution. In its conclusive form, the role of intuition is to condense--again--in a global, compact view, an analytical solution previously obtained. In this form, too, the role of intuition is to prepare action. That final, concentrate interpretation is destined to make the solution directly useful in an active, productive thinking process.

Before continuing, one remark is necessary. When speaking about action--in connection with intuition--we are referring to both external and intellectual actions. The external ones are the primitive correlates of intuitions. Direct spatial and time estimations, rudimentary forms of distance evaluations and of relative frequency evaluations, and global comparisons, are examples of this type.

On the other hand, while thinking, while mentally experimenting, while elaborating an explanation, or while looking for a new theoretical model, we are also acting. In this case, as well, we need "an intuitive view" to inspire, to guide, to direct, to prepare the mental action--to keep the mental process moving in a productive direction.

The essential function of (intellectual) intuition is to be the homologue of perception at the symbolic level--with the same task as perception: to prepare and to guide action.

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After the emergence of the elementary conceptual structures in the child (at approximately age 6) and before the appearance of operations (age 6-7), intuition is the main form of a child's thinking. The child does not possess the operational schemata (characterized by composability and reversibility) which are the basic conditions for logical, analytical thinking. Consequently, the child thinks at this stage by restoring to global, half-articulated configurations in which the direct, impressive features have an essential role. At this stage, intuitions are not only particular moments of the process of thinking, they are the process of thinking itself. For that reason, Piaget terms that stage the "period of intuitive thinking."

Though described by Piaget as being specific to the period of intuitive thinking (4-7 years of age), all the features of intuitive thinking are, in fact, common to all of the various forms of intuitions which may be encountered at all age levels.

In other words, intelligence does not abandon its intuitive form when operations appear. Intuitive knowledge is not an immature, transitory form of thinking. Rather, when operational thought appears, intuition continues to survive as a complementary form of thinking. At operational levels of thought, intuition functions as an effective form of cognition, better adapted to action than analytical discursive, time-consuming, logical knowledge.

The above hypothesis has a direct, important implication. If intuition is a condensed, practically adapted version of some information, solution or interpretation, then intuition also has something to do with previously acquired experience. This does not exclude the possibility of a priori schemata intervening in our representations of facts. But it is natural to assume that if intuitions prove themselves to be useful mental tools--adapted to the particular requirements of certain classes of situations--then they must be the result (totally or partially) of personal practical experience.

Intuitions and Mental Skills

If intuitions are the product of experience, then are they merely well-established mental habits? Is there anything more in an intuition than in any well-structured, mental system of skills?

Intuitions cannot be reduced to mental habits. A habit is essentially a stabilized manner of acting in response to a class of situations. An intuition is primarily a cognition, i.e., a subjective reflection--correct or not--of some real facts. The novelty of this kind of cognition is that it is adapted, by its features, to the needs of action. Although intuition is, generally, a derived form of cognition, like conceptual knowledge, it has the features of being immediate, inner-structured, self-evident and coercive--all of which are characteristics of perception. Being like perception, and as a result of their common qualities, intuitions have the role of preparing and guiding action. So, intuition is generally shaped by practice, in connection with a defined category of situations. But the product of that experience is not merely a mental--or practical--habit. It is, rather, an image, an interpretation, an evaluation, which, by its intrinsic nature, can be directly and immediately translated into terms of adapted reactions. (Recall that by the term action is meant both external and mental forms of activity).

Let us take a few examples:

Consider the formula for solving quadratic equations. Knowing the formula, possessing the corresponding skills for using it automatically, does not generate any intuition. So, intuition is not merely a well-established mental habit. In other words, intuition is not reducible to an algorithm. It must be something more.

For a different example, consider the formula for calculating the number of permutations of n elements. The formula is $P(n) = n!$ With one element, there is, of course, only one possibility. With two elements there are two possible permutations. This is intuitively evident. Now add a third element. Each of the two previously obtained permutations AB and BA, will provide three permutations because the third element, C, may occupy three different places:

CAB, ACB, ABC, CBA, BCA, BAC

The representation of how permutations can be built when adding more and more elements is an intuitive representation. When adding a fourth element, each of the above obtained permutations will provide four permutations because there are four different places for introducing the fourth element.

Intuitive understanding is sufficient for reaching, by extrapolation, the general formula: $P(n) = 1 \cdot 2 \cdot\cdot 3 \cdots n = n!$ Is this formula a mere algorithm or does it express something more? After realising how the formula was deduced, we reach a level and a kind of understanding which is beyond the simple knowledge of a programme of action. The process of successively multiplying by 2, by 3, by 4, etc. expresses directly the way of building the permutations. The formula $P(n) = 1 \cdot 2 \cdot\cdot 3 \cdots 4 \cdots n$ contains in its specific apparent structure its justification. When recalling the formula, one recalls in a direct, global manner the basic idea of how permutations are produced as a function of the number of elements, and why they are produced in that manner. This is more than a mental skill, i.e., more than a simple programme of action; it is grasping the meaning of a process with an extrapolative perspective in mind.

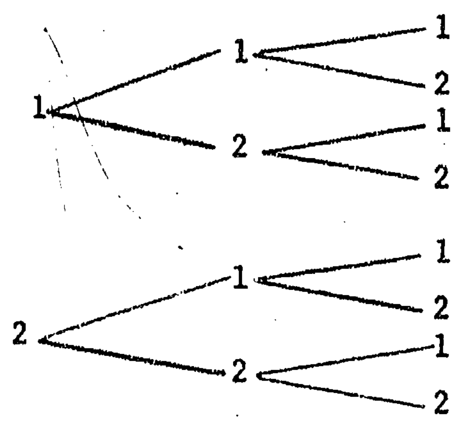
Briefly speaking, intuitions are mental structures based on previously accumulated experience and expressed in an interpretative and predictive form of knowledge. Therefore, intuitions are generally based on systems of mental skills. But mental skills do not, by themselves, entirely explain the particular nature of intuitions. Accumulated experience, in a given field, must also be expressed in an interpretative and predictive global view. Some kind of induction, largely performed unconsciously, is probably the main source of that global representation. Intuitive acceptance involves a feeling of obviousness which is the effect of congruence between justification of an interpretation and a programme of action.

Intuition and Images

A second aspect which has to be taken into account refers to the role of images. Visual images (graphs, schemata) are often used to provide mathematical statements with an intuitive dimension. However, an image is not an intuition by itself. In order to be an intuitive way of understanding, the image has to be included in an active process. The role of the image-intuition is therefore a double one: (1) to unify, or synthesize, information and (2) to prepare, guide, or anticipate action (on the basis of that information). Therefore, a visual representation will be able to contribute efficiently to an intuitive understanding of some mathematical truths if it can suggest the dynamics of the corresponding intellectual process.

Return to the example of combinatorial procedures. Suppose we want to use a tree diagram for solving a combinatorial problem--for instance: How many numbers of three figures could be obtained from two given figures: 1 and 2?

The corresponding image is:



Now, the question is: Is that static image sufficient to elicit an intuitive understanding of combinatorial procedures?

The question is trivial and the answer is evident. For a solid intuitive understanding, the pupils must participate in the building process. That is, if the tree diagram is included in a building process and if it symbolizes a building process, then it will be interpreted as a programme of action, and thus will gain intuitive efficiency.

Let us take a different example: an intuitive concept of geometrical locus. To teach such a concept, a teacher could present the pupils with some drawings: a circle, the perpendicular drawn on the middle of a segment, an angle and its bisectrix, etc. These can be used as examples of the concept of a geometric locus. Moreover, they are represented by visual images. However, such static images contribute little to an intuitive understanding of geometrical locus. They lack a constructive, dynamic component. The circle, the bisector, etc., must emerge as a Gestalt from the set of points which have been drawn by the child himself, in conformity to a given rule. The formal proof comes afterwards. But the intuition of a specific geometrical locus (and, by way of generalization, of the concept of geometrical loci in general) can be obtained only by joining, in one single "vue d'esprit" the deductive and the constructive, the iconic and the enactive aspects of the concept.

The graph representing a function is not only an image, an iconic translation of a concept, it is also an essential conceptual aid due to its intuitive features. It represents in a unique, condensed view, the dynamics of a mathematical relation. It suggests not only a limited situation (for instance, as a result of the variation of x between a and b), but the trend of the whole process.

Intuition and Experience

We thus come back to one of our basic assumptions: intuitions have to play a constitutive role in an active demarche; but they can only be elaborated in the course of such an active approach. By way of merely formal, verbal explanations, it is possible to teach only a conceptual structure. Such a structure may appear to be very convincing from a formal, logical point of view ("I am now convinced that this must be so, because I do not find any lacuna in your proof"). But the learner may add: "In spite of this, I do not feel completely comfortable with your statement."

In order to overcome such a conflict, i.e., in order to create a genuine, intuitive acceptance of the statement, both the proof and the statement must be given a behavioral dimension. Such a behavioral dimension can be created only if the student has the opportunity to be personally, practically, and experientially involved in the process.

Probability is an area in which students often lack a good intuitive background. It is in this domain that the role of experience in creating new, correct intuitions can best be shown.

For Piaget and Inhelder (1951, 1975), the concept of probability can be understood by the child only at the formal operational level, as a synthesis between chance and deductive operations. In fact, that synthesis does not generally occur spontaneously at the formal level. My explanation is that current science education is almost completely oriented toward elaborating deterministic forms of thinking and interpreting. Consequently, a lack of equilibrium appears between the two components. As an effect of actual school education, students will assimilate various mental skills and intuitive forms of interpretation, helping them to cope successfully with deterministic situations. On the other hand, when facing (even very simple) problems referring to random events, students will frequently give wrong solutions. They are lacking not only the corresponding technical procedures, but first of all, basic, correct mental intuitions--what Freudenthal (in this same monograph) calls "mental objects."

Let us take an example: In an attempt to teach twelve-year-old pupils the concept of probability, a teacher began with the concepts of possible, certain, and impossible events. Next, the teacher gave various examples of certain events and chance events--using boxes containing colored marbles, e.g., white and red marbles. For example, from a box containing an equal number of red and white marbles, the pupils had to draw a marble at each trial (with replacement). The experiment was performed by a group of 5 pupils--each of them performing 20 trials. After a hundred trials, the record was 47 reds and 53 whites. The teacher concluded: "As you can see, the number of white and the number of red marbles drawn are very close (47-53). They both are close to 50. What do you think: Will we get similar results when replicating that experience? In other words, if a different group of pupils repeat the experience in the same conditions, will we again get an approximately equal number of white and red marbles?" The pupils' answer was negative: "No, we cannot predict anything because the outcomes are random." What is lacking is exactly that synthesis between the possible and the necessary which characterizes the scheme of probability. This is not a matter of a computation procedure. This is a matter of "mental objects." The pupil is not yet prepared to overcome the contradiction between these two opposed categories of events (certain and random) and to understand the possible rationality, the possible predictability of mass phenomena. In Piaget and Inhelder's view, probability is one operational schemata which (together with the schemata of combinatorics, proportion, etc.) characterizes the formal operational period.

I do not exclude the possibility of describing the concept of probability as an operational schema. I submit, however, that the capacity to understand probability and to correctly use probabilistic procedures requires some specific intuitions. The basic intuition is that of the possible regularity of change events when considering them as mass phenomena.

Let us return to the basic hypothesis: Intuitions are shaped by direct involvement in a practical experience. <u>To create probabilistic intuitions in pupils, there is no other way than by helping them experience probability</u>. For instance, if children, organized in groups, perform the same random experience in similar conditions, they would realize that, while the single outcome of each trial is unpredictable, <u>sets of outcomes follow some regularity</u>. As the number of trials increases, the outcomes tend to distribute themselves according to certain predictable proportions.

In a recent experiment (Fischbein, 1975), subjects were asked to determine the probability of getting the pair 5-6 when throwing a pair of dice. The pupils (grades 6, 8, and 10M (classes of mathematics)) had been previously taught the multiplication rule for probabilities. They knew, for instance, that P(6,6) = 1/6 x 1/6. The subjects did not see any difference between the probability of getting a pair of equal numbers (for instance: 6,6) and the probability of getting a pair of numbers which were different (for instance: 6,5). Twenty subjects were examined at each level. The frequencies of correct answers, according to grade, were as follows: grade 6: 1; grade 8: 2; grade 10: 4; grade 10M: 3. We concluded that generally subjects do not possess natural intuitions for estimating compound events. Furthermore, in order to create such support, a verbal explanation and lists of possible outcomes are not sufficient. The subject must participate in the random experiment. He must see, for instance, that the relative frequencies of pairs of equal numbers are in fact smaller, as compared with the relative frequencies of pairs of non-equal numbers. Some subjects can perform such an experiment mentally, but this is usually not the case. Moreover, as a result of participating in real experiments, subjects usually improve their capacity for conducting such mental experiments.

This problem involves the following aspects:

(a) the training of a general capacity to perform mental experiments by using conceptually controlled images;

(b) the training of subjects in a certain domain. They get used to that domain, to its specific objects, properties, and phenomena and can then mentally perform the required manipulations.

(c) the role of age. We can suppose that formal operational subjects are better adapted in conducting mental experiments than younger subjects, provided that they have received previous practical training in the respective area.

The main point is that intuitions can be elaborated (or corrected) only as a result of personal involvement in a practical experimental activity. An intuition is not a passive copy (iconic or symbolic) of a given reality. An intuition is always a construct, an interpretation, a presumptive explanation, a guessed solution. An intuition is not a perception and not a simple substitute for a perception-like elementary mental image. An intuition is a theory presented in a perceptual-like manner; it is characterized by globality and practical effectiveness. Such a symbiosis between a theoretical model (with its capacity for interpreting, explaining, and predicting) and perceptual qualities (globality, obviousness, imperativeness, effectiveness) can only be created in a practical, experimental activity. Such an activity requires moments of thoughtful guessing, of assimilating and coordinating information, and of formulating plausible predictions. The result is a mental structure having both the qualities of a theory and the qualities of a perception. Like a theory, it is an explanatory or predictive device which is able to connect in one structure the variety of facts gathered; and, like a perception, the same conclusive construct will possess the qualities of a perceptual representation (as a result of the pragmatic character of activity itself).

Including the student in a process of inquiry in order to create intuitions perfectly suits the teaching of probabilities, especially at initial, introductory levels. Thus, the problem in teaching probabilities is primarily the problem of forming an adequate intuitive background. The formulas used for solving probability problems have a relatively simple algorithmic structure. However, even simple probability problems require specially adapted intuitions which are generally absent if the student has not received special training. This is not the case with elementary geometry where most of the concepts and operations have a natural intuitive correspondent.

In order to develop adequate probabilistic intuitions, there are a variety of experimental situations: throwing dice, tossing coins, extracting marbles, playing roulettes, etc. All these activities can be performed in a practical, experimental way. The pupil has to analyze a given situation, make predictions, organize an adequate experiment, watch, record and classify outcomes, compare results with predictions, etc. Various types of distributions (binominal, normal, etc.) receive an intuitive backing by effectively performing the related experiments (and not just seeing the graphs of the corresponding functions).

Is the Intuitive Acceptance of Mathematical Statements Necessary for Mathematical Education?

Intuitive modes of representation and identification range over a wide spectrum of apparently very different mental structures. At one end are the most elementary forms of intuitive knowledge, almost reduceable to perception itself: the estimation and decision-making sensory-motor structures, like those which are expressed in space and time evaluations. At the other end are the complex, sophisticated intuitions of scientists and mathematicians which are almost ready to be translated into fully formalized analytical presentations. Between these two extremes lie a large variety of intuitive representations, interpretations, explanations, more or less connected with figural models and verbal forms of expression. They all share the same basic features: they all synthesize a large amount of personal experience on a global, self-evident extrapolative vision.

A beautiful example is that of the discovery by Galileo of the law of inertia described by Wertheimer (1945). An outline of the story follows:

The following statement seems to be common-sense:

"A moving body sooner or later comes to a standstill if the force which is pushing it no longer acts. Isn't that true? It is obvious." (Wertheimer, 1945, p. 161)

The preceding situation involves an elementary intuition. It is based on all our experiences concerning moving bodies. It is a theory: it is a general, extrapolative interpretation of a class of facts. We feel entitled to say that it will always be so. The theoretical justification may sound as follows: movement always consumes energy. On the other hand, that theory appears to be--in the common-sense form--a global, completely trustful view. I can safely predict that, after throwing a body under any possible circumstances, the body will stop moving sooner or later, provided nothing else intervenes. I know it, I feel I am completely sure of it, I do not need any further explanation. It is so. It is an intuition.

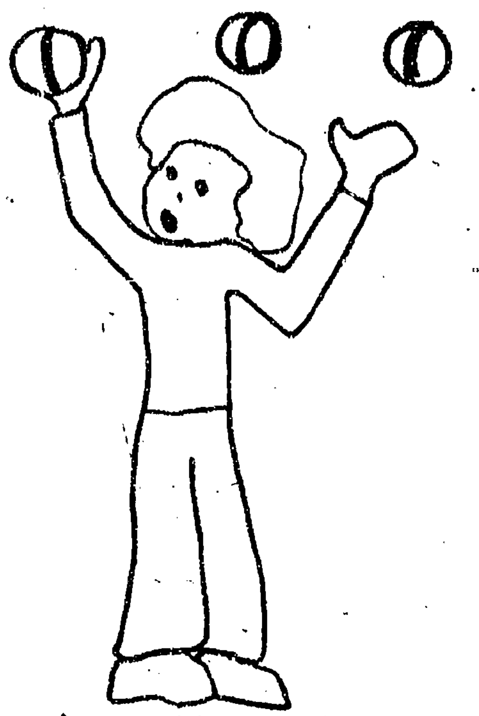
At the other end of the spectrum are Galileo's intuitions about motion. He carried out a series of experiments with free-falling bodies and with rolling balls on inclined surfaces. He stated that acceleration decreases consistently with the angle of inclination. Wertheimer continues:

> Then suddenly he asked himself: "Is this not just half the picture? Is not what happens when one throws a body upward, when one pushes a sphere in the uphill direction, the symmetrical other part of the picture which repeats like, a reflection in a mirror, what we already have and which completes the picture?"
>
> When a body is thrown up, we have not positive but negative acceleration....But does this complete the picture? No. There is a gap. What happens when the plane is horizontal, when the angle is zero and the body is in motion? (p. 161)

The logical conclusion is: if the acceleration is zero, there are no changes in velocity. The ball will continue to move constantly if no external force intervenes: "A body moving at constant velocity will never come to rest if no external hindrances are at work...." (p. 49)

"What an amazing conclusion!" writes Wertheimer, "apparently contradicting all familiar experience, yet required by the constancy of the structure" (p. 164). As a result there is another theory--this time one based on a logical analysis--which contradicts the first, though it was deeply rooted in our life experience. Can such a theory be transformed into an intuitive acceptance (i.e., as a self-evident truth)? And, is it necessary that such a logically based theory should be associated with an intuitive feeling of obviousness, of direct credibility? Our answer to the second question is: Yes, particularly if it must be opposed to a different, incorrect intuition. Why? Simply because for non-standard questions, there is a high probability that the student will answer according to his intuition and not the learned conceptual framework.

Laurence Viennot (1978), a French physicist, presented the following problem to high school and university students. Figure 1 represents a set of balls which are being juggled by a juggler. The images of the balls are frozen at a certain moment of their flight. All the balls are supposed to be in the position, but have different speeds and directions of motion as indicated in the figure. The question is asked whether the forces acting



on all the balls (or masses) are identical at the instant shown (air resistance considered negligible). The correct answer is that the forces are equal. The only forces which have to be taken into account here are the weights of the bodies. The bodies being identical and at the same heights, the forces are equal. However, most subjects consider the forces to be different. (For instance from 49 students in physics in the third year of University, 37% have affirmed that the forces are equal, 55% that the forces are different, and 8% did not reply). Analyzing the subjects' explanations it becomes clear that the students who thought that the forces were different were using the following (intuitive, non-explicit) interpretation: a ball which has been launched upright keeps rising because it has been given an impulse and that impulse has not yet been used up. Consequently the bodies--though being at the same distance from the earth--but having previously covered different distances in their motion, are possessing different "capitals of forces" (a student's formulation quoted by L. Viennot) (Viennot, 1978, p. 19).

Intuitions are often very coercive and persistent. Consequently, wrong intuitions may have a misleading influence even in persons possessing a good theoretical preparation in a field. This phenomenon is well known in probability. Furthermore, such misunderstandings and errors may be found in other branches of mathematics as well.

Consider the following example. "C is an arbitrary point somewhere on segment AB. We divide and subdivide segment AB by two, by four, etc. indefinitely. Will we arrive at a situation such that one of the points of division will coincide with point C?" The question was put to junior high school pupils (grades 5 to 9) and, in a non-formal manner, to university students (in mathematics). About 80% at all grade levels (including university students) answered that as the process of division is not limited, the point C will coincide--sooner or later--with one of the points of division. In grade five, 81.2% and in grade nine, 88.1% answered that way. Yet, in grades 8 and 9 (and in some cases even before) the pupils have learned about rational and irrational numbers (Fischbein et al., in press).

Our main explanation of these results is that, <u>intuitively</u>, infinity is equivalent to non-exhaustible. For intuition, the various "degrees" of infinity do not exist. Consequently, as an effect of the non-limited process of division, each of the points of the segment may be reached, sooner or later. Although the grade 8 and 9 pupils possessed the conceptual prerequisities for a correct answer (for instance that the possible coincidence depends on the position of C), they generally gave a wrong answer following their intuitive bias.

The problem is not only to avoid the negative effect of some immature or false intuitions. Correct intuitive representations, interpretations or explanations represent essential prerequisities for mathematical thinking and for scientific thinking in general. Michael Pollany expressed this as following:

> ...We can understand mathematics only by our tacit contribution to its formalism. I have shown how all the proofs and theorems of mathematics have been originally discovered by relying on their intuitive anticipation; how the established results of such discoveries are properly taught, understood, remembered in the form of their intuitively grasped outline; how these results are effectively reapplied and developed further by pondering their intuitive content; and they can therefore gain our legitimate assent only in terms of our intuitive approval. I have indeed shown that all articulation depends on a tacit component of the same kind for conveying a meaning accredited by the person uttering it.
> (M. Polany, 1958, p. 188)

In fact--as has been frequently repeated--we must distinguish between the axiomatic form of a constituted branch of mathematics and mathematical thinking as a productive process. Mathematical thinking is a constructive activity during which we try, we combine, we guess, we formulate assumptions, we check, we extrapolate, and we make large mental jumps. Mathematical activity possesses all these qualities which are shared by every adaptive intelligent activity. Therefore, if we admit that intuitions are a sine-qua-non component of intelligent behavior, we must also admit that mathematical thinking normally includes intuitive ways of looking for, of trying, of checking, and of representing.

If we refer to anticipatory intuitions, things look rather trivial. Everybody agrees that while striving to solve a mathematical problem the full solution is generally anticipated by a global view of it. However, it is less evident that an intuitive understanding, an intuitive version of mathematical truth, is generally no less important for mathematical activity than intuitive anticipations are.

Productive mathematical thinking necessarily includes--as an active component--intuitive forms of acceptance, representation, and proof. The crucial point here is not to replace the formal structure by intuitions but, rather to inject it (the formal structure) with the specific dynamics of human thinking. In other words, while mathematical truth is guaranteed by logical schemes, the progress of mathematical thinking is stimulated and guaranteed by its various possible intuitive models and representations. Recall the definition of continuity of a function $f:E \to R$ in a given point x_0 of E.

A function $f:E \to R$ is continuous at a point x_0 of E if, for every neighborhood U of $f(x_0)$, there is a neighborhood V of x_0 such that for any $x \in V$, we shall have $f(x) \in U$.

There is here a string of concepts the meaning of which is not easily grasped. Presumably, each of the concepts has been previously clearly defined and the student knows these definitions. But it seems to be a difficult task to coordinate them in a unitary, purely logical meaning.

In a more primitive intuitive interpretation, the same idea could be expressed as follows: A function $f:E \to R$ is continuous in a point x_0 of E if, for values of very close to x_0 we could find values of $f(x)$ which should be as close as we want to $f(x_0)$. That is, we should be able to approach the $f(x_0)$ value as much as we want, by $f(x)$ values. The primitive idea is that of points which may be considered--or not--as constituting an intuitively (visually) continuous line.

Such primitive interpretations of continuity may be dangerous, because they might distort mathematical thinking. Illegitimate extrapolations may be made if intuitions are permitted to invade mathematical activities. On the other hand, full trust on intuitions will sharply limit the freedom of creative mathematical thinking.

From such considerations should we conclude that the intuitive interpretations must be banished systematically from mathematical thinking? Surely not, for the simple reason that <u>without intuitions we cannot think productively</u>. Pavlov is reputed to have said: "Facts are the air of the scientist." But there are not only physical, material "facts." When thinking creatively we are necessarily using "mental facts," "mental objects." Intuitive representations are the most stimulating category of such mental facts. An intuitive interpretation has the capacity to inspire, to guide, to elicit, and even--sometimes--to check productive adaptive activity. The superiority (and the danger) of intuitions is that they do not offer a merely phenomenal information. An intuition is a <u>theory</u>, expressed in an elaborated and condensed cognitive structure and based on previously lived, personal, more or less generalized experience.

The intuition of continuity is not merely the <u>image</u> of a continuous line. Some implicit affirmations are contained in that intuition. We may try to make some of them explicit, i.e., there are no holes, no interruptions, no "absences" in the "object" which is considered to be continuous. This also implies the idea of infinite divisibility of that continuous object and the idea that every part of it is connected to other parts of it, etc. All these ideas appear rather confused when we try systematically to derive them from the primitive intuition of continuity.

Fischbein

Our intention should not be to make an inventory of such confused implications. We wish only to demonstrate that even this rather rough representation of continuity is not reducible to a pure image: it is a theory (or a cluster of non-explicit theories). Its specificity with regard to a formalized theory is its "compactness," its rather perceptual mode of manifestation.

While thinking about continuity, we usually join to the formal definition some compact, global representation of it, which must not necessarily be that of the beginner. It may be more defined, an improved version which fits the conceptual prescriptions of the mathematical definition better (this is a "secondary intuition").

The specificity of an intuitive representation does not lie in its primitiveness or roughness. The specificity of an intuition is defined by its globality and immediacy. Corresponding to the various levels of mathematical abstraction there may be various modes of intuitive interpretations of the same concept. In such a global, self-evident representation there are usually mixed images and verbally expressed interpretations. Beyond all these, there is a key unifying meaning, inspiring, directing, stimulating and controlling the mental constructive process.

Poincaré wrote:

> Prenons, par exemple, l'idée de fonction continue. Ce n'est d'abord qu'une image sensible, un trait tracé a la craie sur le tableau noir. Peu à peu elle s'épure; ou s'en sert pour construire un systeme compliqué d'inégalités, qui reproduit toutes les lignes de l'image primitive; quand tout a été terminé, on a décintré comme après la construction d'une voute cette representation grossière, appui desormais inutile, a disparu et il n'est resté que l'edifice lui même, irreprochable aux yeux du logicien. Et pourtant, si le professeur ne rappelait l'imate primitive, s'il ne retablissait momentanement le cintre, comment l'élève devinerait-il par quel caprice toutes ces inégalités se sont echafraudées de cette façon les unes sur les autres? (Poincaré, 1914, p. 134)

A mathematically formalized truth would appear as a strange, arbitrary combination of statements without a basic intuitive representation serving as justification for unifying the statements in that manner and not in another.

What is said with regard to the learner is equally true for the creative mathematician:

> Pour le géometre pur lui même, cette faculté est necessaire, c'est par la logique qu'on démontre, c'est par l'intuition qu'on invente. Savoir critiquer est bon, savoir créer est mieux. Vous savez reconnaitre si une comninaison est correcte; la belle affaire si vous ne passedez pas l'art de choisir entre toute les combinaisons possibles. (Poincaré, 1914, p. 137)

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We are acting mentally on intuitions with intuitions, according to the general logical rules. Lakatos wrote:

> Now this bleak alternative between the rationalism of a machine and the irrationalism of blind guess does not hold for live mathematics: an investigation of informal mathematics will yield a rich situational logic which is neither mechanical nor irrational.... (Lakatos, 1963, p. 15)

> ...informal, quasi-empirical mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations. (Lakatos, 1969, p. 61)

> ..None of the "creative" periods and hardly any of the "critical" periods of mathematical theories would be admitted into the formalist heaven, where mathematical theories dwell like the seraphim, purged of all the impurities of "earthly uncertainty"....On those terms Newton had to wait four centuries until Peano, Russell and Quine helped him into heaven by formalizing the calculus. (Lakatos, 1969, p. 3)

David Hilbert adds:

> Who does not always use, along with the double inequality $a > b > c$, the picture of three points following one another on a straight line as the geometrical picture of the idea "between"? Who does not make use of drawings of segments and rectangles enclosed in one another when it is required to prove with perfect vigour a difficult theorem on the continuity of functions or the existence of points of condensation? Who could dispense with the figure of the triangle, the circle with its center or with the cross of the three perpendicular axes? Or would give up the representation of the vector field or the picture of a family curves or surfaces with its envelope which plays so important a part in differential geometry, in the theory of differential equations, in the foundations of the calculus of variation and in other purely mathematical sciences?

> The arithemetical symbols are written figures and the geometrical figures are drawn formulas. (Cf. Reid, 1970, p. 79)

If, at the beginning of the above quotation, we would suppose that, for Hilbert, these images have only a mnemonic role, the last statement casts a different light on his standpoint: "the geometrical figures are drawn formulas," as he said. They are symbols of concepts or of mental operations. In fact, they are more than simple, pure conventional signs: they have a key role in suggesting, orienting, organizing, and prompting our ideas and in creating a feeling of compact meaningfulness, of inner structurality of the corresponding mental fact.

Fischbein

These intuitive representations are not heterogeneous to mathematical thinking: they are a necessary part of its dynamics. The condition is that these iconic symbols should be controlled by the corresponding formal structure. The use of geometrical symbols as a means of strict proof presupposes the exact knowledge and the complete mastery of the axioms which lie at the foundation of these figures; and, in order that these functional figures may be incorporated in the general treasure of mathematical symbols, a vigorous axiomatic investigation of their conceptual content is necessary.

According to Hilbert, the founder of modern axiomatics, the living process of mathematical thinking must not and cannot be "purged" of intuitive representations. Consequently, the great problem of mathematical education is not how to avoid the interference of intuition in the flow of mathematical thinking. Such an avoidance is not a realistic enterprise. Rather the problem is how to control, conceptually, intuitions without stifling them. If good intuitions are lacking we have to build them. If wrong intuitions are present, we have to eliminate them. If vague or distorted intuitions are present, we have to correct them, to clarify them and to include them into a conceptually controlled process.

There are various ways for improving and enriching the intuitive side of a concept or a statement. Some of them can be deduced from the former examples. I would like to emphasize only one aspect which seems to me of high didactical value. I am referring to the act of elaborating a conceptual, rigorous structure (for instance, a definition) corresponding to some intuitively known property or operation. This is what creative mathematicians are usually doing, but it is very seldom that pupils are also asked to do by themselves. The new concept must not be built aside from the primitive representation, but in connection with it, in a dialectical dialogue with it. The new concept should inherit the inner structurality, the meaningfulness, the quality of objectual compactness, expressed by the original intuition. On the other hand, the intuitive representation itself will gain in clarity and in precision in communicability.

Let us come back to the problem of continuity. We know intuitively what continuity means. For instance, we can draw a continuous line on a sheet of paper. Let us imagine that the line is the graph of a function. We would like to make explicit what we mean by continuity. Let us try to express the formal condition defining completely and exactly, without any ambiguity, what a continuous graph means and consequently what a continuous function means. This is made step by step. I discover that the first problem is to define the continuity in a given point. For the primitive intuition this is a new problem because continuity is basically connected with a structural, Gestalt, representation. In order to formulate the idea of continuity I have, first of all, to destroy continuity itself, to smash it into infinitesimal fragments. The question ends up worded as follows: Is it possible that for values of x close enough to x_0, to find values of $f(x)$ which should be as close as we want to $f(x_0)$? At the next step, I return to the function as a whole by generalizing the definition of continuity in a particular point. This overall approach is probably very common. First, the analysis of the concept in reference to the particular paradigm or element of the set is considered. Second, an exact description of the properties is obtained. Third, the set as a whole is reconsidered, etc.

What follows from such a constructive process is the inclusion of the primitive intuition in a rigorous conceptual framework. But the intuition itself is no longer what it has been. The intuition has been re-built and transformed into a quasi-new mental structure in which some echoes of the previous ideas of continuity have been preserved but in which the essential is represented by the mathematical structure itself, seen as a whole.

What is important here, for building or rebuilding an intuition, is primarily the constructive process. For reaching an intuitive acceptance of a mathematical statement (definition, theorem, formula) that statement must be elaborated by the learner himself as a result of a personal--even original--search effort. The didactic utility of such an active approach is well-known and it sounds rather like a trivial request. What has not been said explicitly is that this requirement refers not to the conceptual structure but to the intuitive component of the understanding process.

Furthermore, the personal involvement of the learner in that constructive process is especially necessary when the respective truth has something to do with an existing, formerly acquired, and stabilized intuition. A new intuition contradicting an old one, or representing an improved version of an old one, cannot be elaborated without taking into account the primary intuition.

Sometimes we arrive at a conflictual situation. For instance, we present a twelve year old child with two segments AB and CD, CD > AB. "How many points are there in the segment AB?" "An infinity of points." "Can we put the points of AB and the points of CD into one-to-one correspondence?" "No." "Why?" "Because the segment CD contains more points." "You said that in both there is an infinity of points." "Yes, but CD is longer. The infinity of CD is greater than the infinity of AC."

Must we avoid such conflictual situations? Surely not. On the contrary, such conflicts must be experienced by the learner himself in order to overcome them.

Let us quote some more answers to a questionnaire referring to the concept of infinity (Pupils grade 10).

"There are more points in a square than in a segment. (I am answering according to my feeling and not to what I have learned)."

"There will be no correspondence between the natural numbers and the points of a line, because the points of a line have no beginning and the natural numbers begin with the number 1."

"There is no correspondence between the natural numbers and the positive even numbers, despite that both sets are infinite, because the ratio between them is 1:2."

It is clear that the students use their "finitiste" schemas, based on deeply rooted intuitions ("the whole is greater than its parts", etc.) in interpreting relations between infinite sets. What is surprising, and very interesting from a psychological standpoint, is that the students accept their own contradictory affirmations without protesting and without looking for a solution. Very probably, they get accustomed during their school years to tacitly accepted interpretations and affirmations which contradict each other.

The example below illustrates a mixture of contradictory intuitions.

The points of a segment and the points of a line can be put into one-to-one correspondence "because a point is a concept which cannot be concretized; that is why we can always find a smaller point and, on a given line, we can make more points." So, in the pupils' mind a point is simultaneously a concept and an object with a variable area.

But, returning to the constructive process, it seems interesting--both from a psychological and a didactical point of view--that, for the intuitionist doctrine in mathematics, an intuition is equivalent to the result of a constructive mental process. An idea is intuitively clear, it is intuitively acceptable, only if it can be effectively constructed by a mental process. In Heyting's words:

> Intuitionist mathematics consists of mental constructions; a mathematical theorem expresses a purely empirical fact namely the success of a certain mental construction. Thus, "2+2 = 3+1" means I have effected the mental constructions indicated by (2+2) and by (3+1) and I have found they lead to the same result. (Wilder, 1965, p. 247-248)

I do not intend to discuss the role of intuitionism in the history of mathematics, or its validity as a mathematical conception. But I am convinced that an analysis of the Intuitionist approach may be very suggestive in the didactics of mathematics. This is not to claim that only those concepts which express a constructive process are mathematically valid. But I must take into account the fact that only those concepts and statements which the learner has attained by his own mental constructive processes have for him an intuitive validity.

A difficult problem is that of mathematical concepts which do not have a constructive nature, such as, for instance, the concept of actual infinity. Such situations generate mental conflicts, hard to overcome. But in such situations, too, didactical means should be invented in order to overcome the difficulty and to attain an intuitive acceptance.

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LEARNING PROCESSES

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The Non-Psychologist View

When I accepted the invitation to present a paper at this conference, I asked: Is it a psychology meeting or an education meeting? I had a good reason to ask this question because I am not a psychologist--not even an educational psychologist. Occasionally, I have thought of myself as an educator but perhaps even this was a misconception. I am a schoolmaster and, I feel obliged to add, a bad one.

I like teaching, and like every schoolmaster, I am proud if my pupils learn what I try to teach. There is only one thing to be even more proud of, that is, the privilege (if it happens) to observe a learning process started by one's mere presence rather than by one's teaching. Good teachers need not think about their teaching, or about that of others, though of course they are allowed to do so. I started thinking about teaching in order to improve it, first my own and later on that of others, in particular of learners of teaching.

To help explain my biases about the situations that I will describe in this paper, there are several reasons for emphasizing that I am not a psychologist. First, I never succeeded in understanding pure theory, or in bridging such enormous gaps between psychological theory and experiment as I noticed, say, in Piaget's work, which other people--psychologists--apparently have not the slightest difficulty to cross.

Second, I would not be able to create the psychologist's laboratory sphere, or to converse with children like psychologists are able to do. I am sure that at my first question children would ask me "What do you mean?" It is a riddle to me--and a matter of admiration--how, say, Piaget and his collaborators managed to interview thousands of subjects without ever being asked this question, not even by subjects who obviously did not understand anything.

Third, I would not be able to have subjects fill out test sheets or react on test questions or be interviewed, without discussing with them their responses. I would want to teach them what is right or wrong and to guide them the right way. It is my feeling that keeping aloof means playing a game against the child. Moreover, I am not interested in what a child or an adult can <u>do</u> at a particular moment but what they can <u>learn</u>.

Finally, I am unwilling to speak about "subjects" when I mean children or adults I worked with.

An Example Involving a Non-Mathematical Concept

One of the children who taught me a great deal about mathematics learning is a boy Bastiaan, born 27 April 1970. Most of my diary (Note 1) on Bastiaan reflects casual observations on his development, made during walks with him from the age of two onwards. It was he who spoiled me and who made me suspicious about children who do not dare ask questions. I always addressed him in adult language, but he never accepted any word that he did not understand. He was patient enough to ask questions as many times as he needed, but he did not show the same patience when conversely I did not understand him.

When Bastiaan did not understand me, he asked: "What do you say, Grandpa?" (This is translated from Dutch; properly used, by adults, it means, "I beg your pardon?") He used this formula up to 15 February 1975. On this day, for the first time, he did not use his old formula, but instead asked for the meaning of a particular word I had used. From this day onwards--he was about 4;10--he always asked for the meaning of a word if there was something he did not understand; he never switched back to his old formula.

This story is an example of what I call <u>observing learning processes</u>. It is observing jumps, discontinuities. It is my belief that learning proceeds by jumps and that, in learning, the only thing you can reliably observe <u>are</u> jumps. With small children these observations are particularly easy, because such discontinuities are often accentuated by an emotional outburst.

The event of 15 February 1975 was an important discontinuity in that particular child's learning life. But, why did it happen just at this moment, and what did it mean in a larger context?

On 7 January 1975 something happened which was related to the event of 15 February 1975. At that time Bastiaan read most of the capital letters, and--globally--quite a few words. He also knew that printed words are composed of letters but he could not yet read by building words from letters. While walking, we came across a parked car of the State Police with the inscription "Rijkspolitie." He knew the car and the inscription. I asked him "read this," and he "read" the word while moving his forefinger from the right to the left. I told him that words are read from the left to the right. He asked me, "And if you read it that way?" "Then it is Eitilopskjir," I answered. He enjoyed this joke and gave me many more words to be pronounced backwards.

The "backwards reading" event seemed related to what happened on 15 February 1975. I cannot ascertain how long before 15 February 1975 Bastiaan knew what words were; long before this date he had asked for meanings of words. However, on 15 February, he discovered a new function of words. Words are parts of speech, and if you do not understand some utterance, the reason may be that some word is unknown. The remedy is to ask what the word means. I do not know if he discovered this strategy at that particular moment. But, at least as far as I know, at this particular moment it happened for the first time that he singled out the word that caused the lack of understanding and he asked for its meaning. And, this was the strategy he would adhere to in the sequel.

Before 15 February 1975, when he asked, "What do you say, Grandpa," I used to produce another version of the same communication. I am pretty sure I never replied "Which word didn't you understand?" It is strange that I did not ask this question because it has been my habit to elicit questions from him by consciously inserting into my talk words I knew he did not know. Was I myself unconscious of the strategy, in cases of non-understanding, of asking for the meaning of a word; or did I not grasp the importance of this strategy or the importance of teaching it to others? Why was I relying on alternative versions, when he did not understand, rather than asking the one question, "Which word didn't you understand?" If I had done so only once, I would certainly have accelerated his learning process. But, honestly, should I have done so?

What general facts about learning can be drawn from this particular learning event in a particular child's life? I do not believe in patterns of development. The number of developments I have had the opportunity to observe is not large, but it is large enough for me to be convinced that they usually are quite different from one another. Children are individuals, and they are learning as individuals. Of course, there are collective learning processes too--learning processes of classes, groups, communities, nations. However, collectives, too, differ from each other, as do their learning processes.

Distinguishing stages and phases is a way to generalize. Making divisions and subdivisions looks like science but history can teach you it is a foregone period of science. It is a cheap pleasure, a diversion from craving for more profound research. Moreover, it is wrong. For example, linguistic psychology knows about three stages of expression: one-word sentences, two-word sentences, and full-blown sentences. I believed this dogma until I observed a girl who started with full sentences, first incomprehensible ones which were gradually understood by her parents and her larger family, and finally, from the age of three onwards by everybody. She never spoke one- or two-word sentences except in cases where adults do so. When I told others about this case, it appeared that it is not as exceptional as I had thought.

What can be learned from particular cases such as the one above or the one involving Bastiaan? They can be used as paradigms. Analyzing one case can provide a model for analyzing others. What is general about learning processes is problems rather than solutions. For example, one particular case can lead to many questions such as the present one, about the various communicative aspects of words.

Some readers may be disappointed by this discussion of one case of learning--particularly because the examples given so far have not concerned mathematics. I must confess that many of my observations which have been most instructive to me have not concerned mathematics but rather, have involved linguistic or general cognitive development. Nevertheless, from now onwards, I will stick as close as I can to mathematics.

Freudenthal

Examples Involving Number

My next story is about number. I do not say: number concept. I know "concepts" is the fashion now--teaching number concepts rather than number, space concepts rather than geometry, logical concepts rather than thinking--but I believe that these trends are detrimental to mathematics teaching.

When I discuss number, I mean the way numbers and number are acquired as mental objects. It is well-known how this works. Children learn counting the way they learn to recite nursery rhymes and to sing little songs--although memorizing the number sequence in the correct order seems to be a bit more difficult, partly because for numbers above 10 or 20, regularities become increasingly complex. Nevertheless, during the early stages of counting, counting does not necessarily mean counting something. Young children may learn to indicate small numbers (representing, say, their age) by lifting a complex of fingers. However, the learning processes linking counting to cardinal number are terra incognita. Discoveries about invariances may play a part in this development though the invariances that seem most important are not the kind considered by the superficial conservation experiments of psychologists. Such psychological theories about how number comes about are firmly contradicted by the next story I am going to tell.

Bastiaan, at the age of 4;3, showed an unusual behavior towards number. Though he knew some number words, he did not count; in spite of all efforts of his mother to teach him the number sequence, he did not undertake anything unless he were sure he would succeed 100%. Yet, without counting he knew small quantities, and was able to estimate reasonably larger ones.

On 12 July 1974, I observed Bastiaan throwing three objects, one after the other, into a ditch while mumbling: one, two, three. "How many did you throw into the water?" I asked him, and he answered "four."

On 16 July 1974, Bastiaan found a paper snake on the street. I asked him how long it was and was astonished that he said "four meter"--I expected "that long." At the housedoor we met his mother and grandmother, to whom I told the story. They, too, tried to estimate the length. Measuring it with my extended arms I said: "Somewhat more than 6 meters, perhaps 6½." Then, Bastiaan: "Let us say seven." "Did he really understand what he seemed to be saying?", I noted down in my diary.

On 13 August 1974, during dinner at Bastiaan's home, Bastiaan was sitting opposite his younger sister at a rectangular table--his father opposite his mother, his grandfather opposite his grandmother. Suddenly, during the dessert of red currants, Bastiaan lifted his spoon in the greatest agitation and said: "So many we are." Indeed there were six currants on the spoon. I asked him, "Why?" and he answered, "I see it so," and then, "two children, two adults, two grandpa and grandma." Possibly the six currants on the spoon formed the same configuration of six as we occupied at the table, but this I could not see.

Set theory prejudices would prescribe us to interpret the relation made by Bastiaan between currants and people as a one-to-one mapping. It was, however, more global, not atomized into elements but structured into groupings--an isomorphism of two structures. It struck me that one of these structures was "family."

On 14 August 1974, in the Park, Bastiaan showed four snowberries on his hand: "So many we are living at home."

Some time later--I do not exactly know when--he started counting.

On 9 September 1974, on the rim of the sandpit in the Park, Bastiaan was building a long row of sandcookies with a mold he had found. It was the first time he had made such a row of objects. He proceeded from the left to the right. I counted whenever he made a new one. They were 18 when he himself started counting from the left: 1, 2, 3, 4, 5, 8. Then he said something like: "I wanted it to be six." I showed him where the next six finished, but his attention was distracted.

Later on we picked elderberries. He carried the bag and said, "six pounds"--seeming to treat "six" as an indefinite number. I asked, "How do you know?" "My mother can weigh it." "How?" "With a balance." So, Bastiaan really did know that weights are definite numbers.

On 15 September 1974, Bastiaan found a hub cap of a Fiat. He said, "I have already got one." "How many do you have now?" "Two." "How many do you need for a car?" "I do not know." "How many wheels does a car have?" "Four." "How many do you still need?" "Four." "No, how many more should you find?" "Two." At the door he calls out to his father: "I must find two more, then I have four."

On 16 September 1974, Bastiaan grabbed four cookies to take home and said, "When the baby comes, I will take five."

On 18 September 1974, Bastiaan gathered chestnuts in the Park. With five chestnuts in his hand, he first said "five" and then counted 1, 2, 3, 4, 5. Somewhat later he had three in his hand. I asked him whether I should put them in my pocket as I had done with the others. He said: "This must become four, there must be one more." He was mastering number but, in general, he still refused to answer questions of "how many" or to count. He counted spontaneously up to six, but I did not know how far he knew the number sequence.

On 6 October 1974, Bastiaan asked "How many is ever and ever six?" He seemed to be asking about repeated addition. I said, "6 and 6 is 12." He asked, "Once more six." I said: 18. He said, "Once more six." I said: 24.

Freudenthal

On 3rd January 1975, I noted that during the prece ng month the boy had mastered cardinal and ordinal number, though sometimes he got confused above 8. He counted non-ordered sets systematically; he never counted the same object twice, and he rarely skipped one. He also counted: one tree, two trees, three trees,.... He also counted mental sets, like airplanes he had gotten at the Institute. Nonetheless, he still did not like counting.

Later, Bastiaan asked: "If you make 12 and 0, how much is it?" My answer: "Do you mean if I put it behind or if I add it?" (He meant the first, which his mother had taught him, but he said the second. He started the talk in order to check whether his mother had informed him correctly.) I told him that 0 is nothing and that if you add it, things do not change. "Over there are zero cats. Can you see them?" "No."

On 27 January 1975, at a landing, Bastiaan reaped a lot of soybeans. "How many did you reap?" I asked. He answered: "100." I continue: "What is more, a hundred or a thousand?" He said: "A million." At this age Bastiaan would count spontaneously up to 29. Yet, he enjoyed "counting houses," that is, walking the streets and reading house numbers. He protested against gaps and he understood the system of even and odd.

Later that same day, Bastiaan said: "When I was 3, Monica was 1; when I am 5, Monica will be 3," and so on with some errors at large numbers. I asked him: "And when you were 2?" He said: "Monica was 0," and he himself continued: "And when I was 1, Monica was in Mom's belly."

On 31 March 1975, Bastiaan said: "Today is the first of April." His father said, "No, that is tomorrow." Bastiaan: "Then it is 0 April today."

Let us stop here. You can read in Bastiaan's experiments on Archimedes' principle (Freudenthal, 1977) how number became more and more objective and developed as a tool to master phenomena. It started, firmly embedded in the child's family life--family and family development is the first structure that is being modeled by number. Counting follows cardinal number; counting invisible sets is a next step--in "Mathematics as an Educational Task" (Freudenthal, 1973) I stressed the developmental importance of counting invisible sets.

I never asked him questions about conservation. It seemed silly to do so in real contexts. Conservation, of whichever magnitude, was never a problem for Bastiaan (Freudenthal, 1977)--implicit knowledge on conservation was rather a source of discoveries to him. For example, at the age of 4;9, Bastiaan conserved the constant difference of two years between his age and that of his sister.

I should add that, with any child I observed, I never met any who had problems of conservation of the type that occur in psychological experiments. I conclude that such conservation difficulties are nothing but laboratory artifacts.

I cannot finish this section without telling an amusing story that happened much later.

28 November 1975 (age 6;7): Walking in a forest, we saw at a distance of about 100 meters away a group of girls pass on horseback. "How many were they?" I ask. Monica answered immediately: "Seven." (This is her favorite number, as Bastiaan's was six.) Bastiaan estimates: "40." I say: "No." He changes to "20." I say: "I think 15 to 20." He: "25." I explain him what "15 to 20" means and (as a joke) I continue: "We can simply count the footprints." He counts from 1 to 20 while stepping from one footprint to a next. I object: "A horse has four legs, doesn't it?" He, with big jumps: "4, 8, 16, (hesitating) 20, 24, (hesitating), 27 (hesitating) 30, 34, (hesitating), 37, 40. It must be 40 horses." I: "But this means there are 10 horses, since one horse has four legs. 10 horses have 10 left forelegs, 10 right forelegs (I am lifting my hands), which make already up for 20." He: "Thus 80; 20 horses have 80 legs."

Examples Involving Geometry

On 13 October 1972 (age 2;5), Bastiaan had his first opportunity to see the labor yards and rowing club (where we regularly came) from the other side of the canal. He recognized and identified all details and he has a good sense of orientation. Once, he wanted to go the straight way to a point that we had always visited by a roundabout walk.

On 6 December 1972, in the Park, I drew a circle around him with a stick and told him he was locked in and could not escape. He accepted it. Only after I had wiped out a little door did he step out.

13 January 1972: He drew a circle around himself in the snow and asked me to do the same, and then to exchange circles--a funny play.

8 April 1974: We were at a public ground, a square meadow, surrounded by a low fence--he asks how a mowing-machine can move in. I show him a gap in the fence at a rather large distance. "This is not big enough," he says. We check it; he is right.

16 March 1975: Spontaneously, Bastiaan "measured" the width of a path by steps. "This is six further." I show him ho I can cross the path in one step. He remeasures the path with two steps and continues measuring distance by pacing.

29 June 1975: Bastiaan and Monica were crossing a meadow, approaching a door to the playground along a somewhat oblique line. We could not see the door because it was hidden by the shrubs. So, Bastiaan went running 10-20 meters parallel to the fence of the playing ground, in order to look straight to the door. He confirmed that the door was open. (See Figure 1 on the next page.)

13 July 1975: Bastiaan looks at a mole-hill and asks, "How big is a mole?" I show him with my hands the length of a mole. "No, I mean how high," he says.

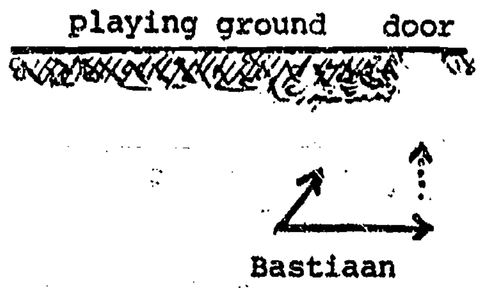

Figure 1. Our Path to the Playground

21 July 1975: At my office, Bastiaan made a long tape full of numbers on the adding machine. He stretches it on the floor of my room. I ask him how many steps it is. He steps along: 11 steps. He has another much shorter tape that is at most three of his steps long. He adds the small tape onto the end of the long one and now measures 17 steps. I do not explain to him that it is impossible. He always counts the 0-th step as number 1.

27 July 1975: On a bridge, I drew Bastiaan's attention to the crooked mirror images of the horizontal and vertical parts of the bridge in the water. I asked him for an explanation. First, he said he did not know, and then he continued: "Because the water is curved."

2 August 1975 (5;3): We were taking a long walk, crossing the Amsterdam-Rhine canal from one bridge to the next--about 2 km. At about one third the distance between the bridges, I ask Bastiaan whether it was half-way. He did not know the word "half," at least not related to distances, so I took a stick and broke it to show him what is a half. He protested, showing me that one piece was a bit longer. I repeated my question about the bridges. He said it was not half-way, but indicated the wrong part as longer. I tried to continue discussing distances but he was not interested. When we were half-way, he spontaneously said, "Here begins the middle." Apparently "middle" was the word he knew, rather than "half."

24 October 1975: While walking, Bastiaan told me that he was preparing to make a record changer, but he still needed a wooden slab. "A square?" I asked him. "No," he answered, "like the front of a car,"--drawing a rectangle on the ground with his forefinger. (He is right, though no adult would describe the front of a car as rectangular.) I said, "A tile is a square. What are two tiles together?" "A rectangle." "Three tiles?" "A rectangle." "Four tiles?" "A square."

31 October 1975: Bastiaan was playing with two irregular pieces of wood. "This is longer and that is shorter," he said, though the difference is small. "What would you say, if there were three of them?" I ask him.

I had to help him with his terms: biggest, smallest, in between, but he was able to apply them to describe family relations involving himself and his two sisters.

1 November 1975: I had collected about 30 chestnut leaf stalks. I threw them on the table and asked him to estimate the number. He hesitated. "Do you think a hundred?" "No," with indignation. "Put them into order." He takes them together in a bundle. "They are not in order." He presses the bundle against his belly in order to find out the longest. I show him it is easier to erect the bundle on the table. The longest one was easily found; then, in rapid succession, he found the longest of the remainder. Soon, the whole collection was ordered according to size.

I cut off stalks of length 10, 9, 8,...,1 cm. He identifies three lengths and names them as such. I cut another stalk of 5 cm. He predicts 5 + 5 = 10 and notes it down.

2 November 1975: During experiments on Archimedes' principle we used the typewritten image of the scale of the spring balance (Figure 2) in order to note down measured weights, and to add and subtract them geometrically. He was not yet able to add weights arithmetically.

When he first looked at the typewriter scale, he protested the use of the letter l as a figure instead of 1. However, he did not notice that the typewritten scale was horizontal rather than vertical and that it was not congruent to the scale of the balance.

25 May 1975: Bastiaan's parents tried to explain to him that an hour has four quarters and that a florin has four quarters--unsuccessfully. I take an apple. "You can halve it this way" (along a meridian), "and once more that way" (along an orthogonal meridian); "these are quarters." Then I asked: "How many quarters of an apple are in the apple?" He answered "eight." "Why?" "You can halve it once more that way" (along the equator), he answered.

16 June 1976 (age 6;2): After a long row of sunny days, Bastiaan noticed clouds. "It will be raining," he said. I explained to him that rain clouds are low and dark, whereas these clouds are high and shiny; no rain will fall out of these clouds. He asks: "How high are these clouds?" I say: "10,000 m." He asks again: "How high are rain clouds?" "1,000 m." He continues: "So, if we are here" (showing to the ground) "and rain clouds here" (showing about 30 cm above the ground), "those cloud are as high as this" (showing a meter above the ground).

Though this was a rough comparison, it was Bastiaan's first explicit showing of proportion--an important event in his development.

16 July 1976: Bastiaan uses a stick with a longitudinal groove together with two bottle tops to represent a machine gun with two bullets. "What does a gun bullet look like?" he asked though he himself knew how

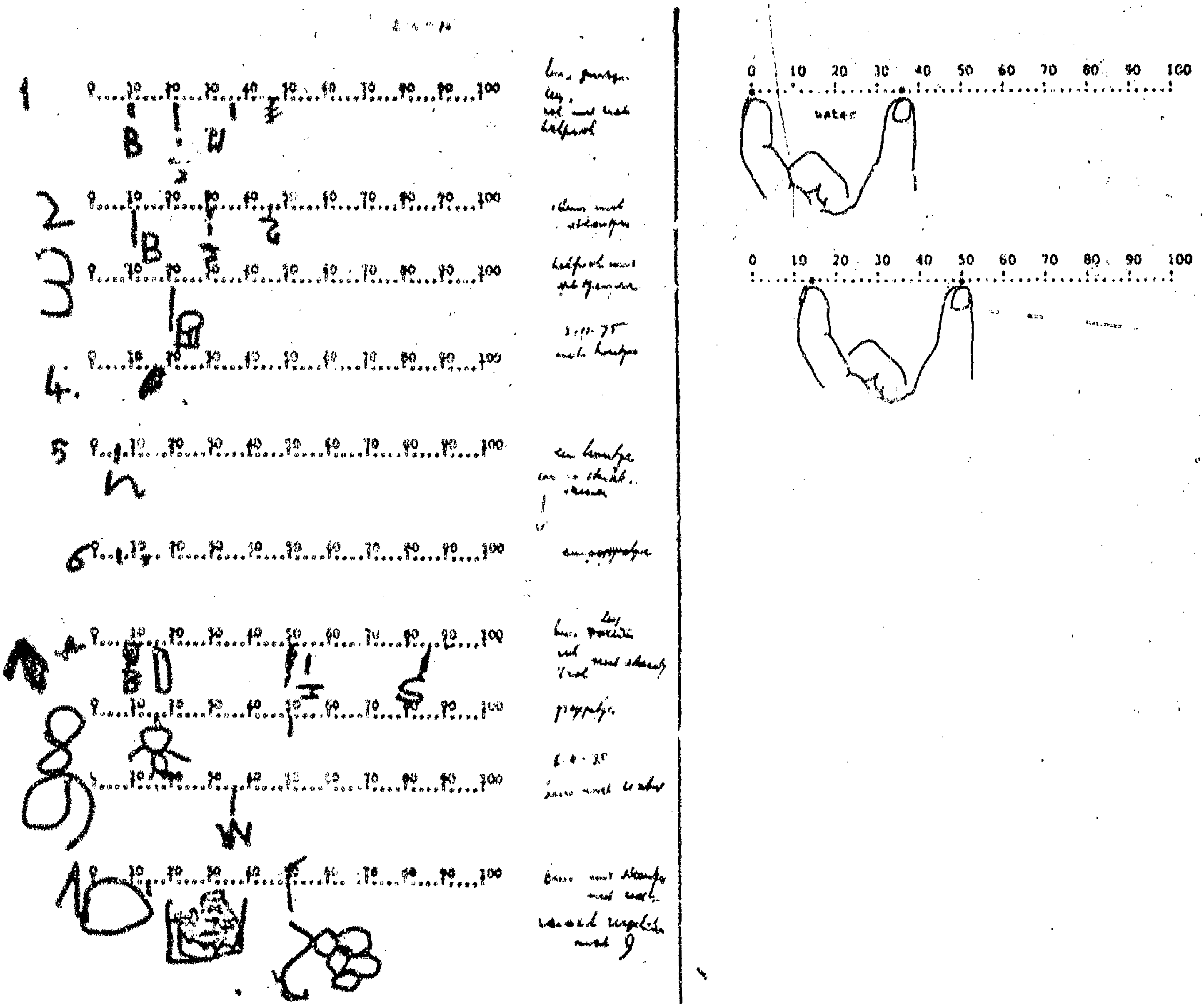

Figure 2. Typewriter Image Spring Balance Scale

to draw it. I add: "A cylinder with a cone on the top." I did not believe he knew the word cylinder, but he only asked what a cone was. I said: "A clown's hat." Then I let him point to cylinders--pieces of tree trunks and trash baskets. I showed him several flat disks. He agreed that they were cylinders though with the reserve: "We shall call them disks."

I asked him what he would see if we cut a cylinder "this way." His answers had nothing to do with the geometrical shape; they were related to the particular cases. I helped him with the word "circle," which he apparently did not know. I showed him examples like the section of a tree, the rim of a trash basket, a button, and he mentioned the sun and the moon. Finally, I showed him a circle-shaped hole in the slip of a beer can. He protested. "This is a bit long." Indeed it was rather elliptical--a difference of less than 10%. The next day he used the word "circle" correctly.

It was most revealing that Bastiaan did not protest about considering the cross-section of a tree as a circle though it was much more irregular than the beer can. But the sharp hole in the can claimed a great precision.

4 September 1976: Bastiaan wanted to make "experiments with centimeters and decimeters." He knew the length of a meter and he knew about the ratio 10:1. I asked him how tall he was. "Better than a meter."

Bastiaan measured his height (122 cm), his arm span, and his step, the ground plan of the suite of two rooms by means of steps--later measuring the whole ground level. After having measured the width of one room, he prepared to do the same with the other. Then, he drew a ground plan on squared centimeter paper with his step taken as a unit (see Figures 3 and 4).

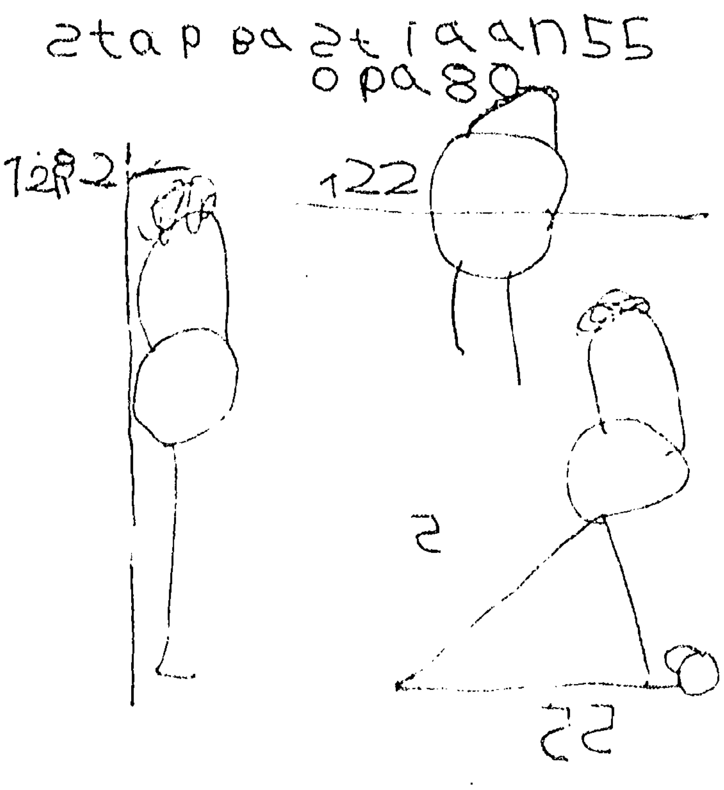

Figure 3. Bastiaan's Drawings About His Height, Step, and Span

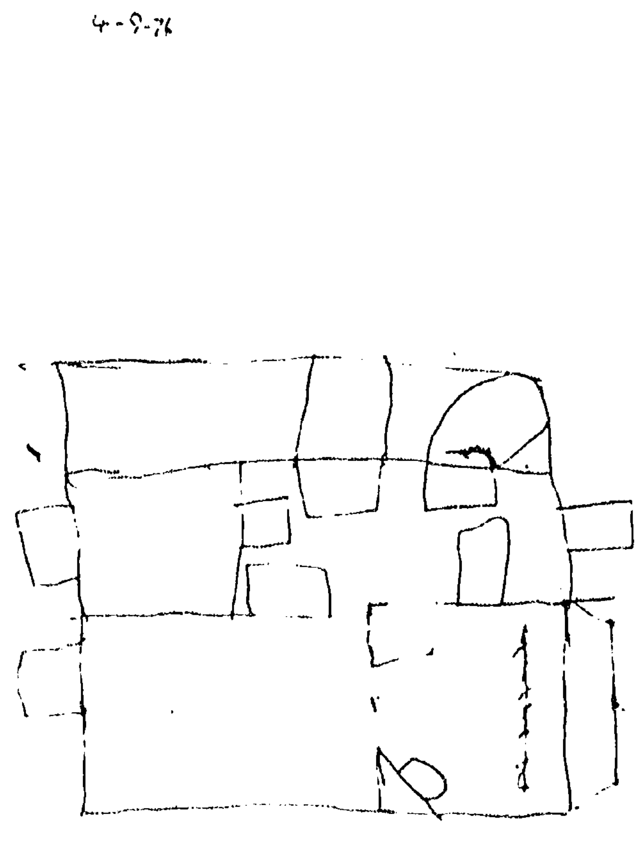

Figure 4. Bastiaan's Ground Plan

The idea of a ground plan was maintained up to the drawing of the doors, which were "turned down."

19 September 1976: At a forked path, after a discussion on cross-roads, Bastiaan said, "There is not only right and left, there is much more, front and back." "How many?" I ask him. "At least twenty."

21 September 1976. During our walk, Bastiaan asked: "Where is the center of the Netherlands?" (Possibly he had heard about Utrecht as such.) I explain to him that this was not easy to determine. Then, I asked, "What is your center?" He pointed to his top. I argued that the center should rather be in his belly. Then, I asked him about the center of a tile or the pavement. First, he denied its existence. Then, he showed its approximate center. I asked him to do it more precisely. He pointed to the groove between the next row of tiles and cut it with an estimated mid-line between the other side (see Figure 5).

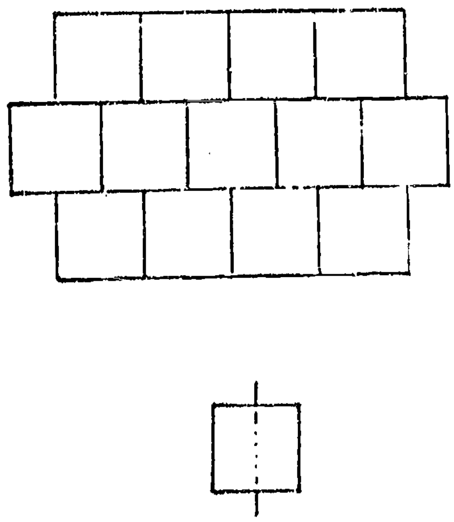

Figure 5. Tiles

I explained to him how to find the center using oblique lines. He drew the diagonals and mentioned the word diagonal. He used this procedure to find the centers of other objects, e.g., a bench.

26 September 1976: Bastiaan found the scraps of perforated sheet iron from which he made his little dogs (see Figure 6).

Figure 6. Bastiaan's Dogs

Freudenthal

27 September 1976: In the sandpit in the Park, Bastiaan had built a large construction of roads, bridges, walls, tunnels. I asked him if he could make a sketch of it when he got home, but he wanted to do it on the spot.

I gave him a little piece of paper. (For a long time, it has been his habit to make written reports.) He said, "I do not have a measuring tape." I replied, "Then, you must estimate." He measured with two forefingers, parallel at a fixed distance--proceeding with the left forefinger in the hole made by the right one. Concerning tunnels, I told him invisible things are indicated by dotted lines.

An ascending dam led him into difficulties. I explained to him what working-drawings are.

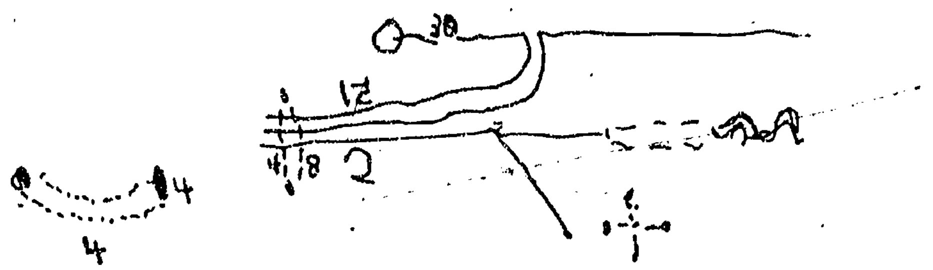

Figure 7. Bastiaan's Map of the Park

A few days later he made a terrific game, a design of roads, with holes at the cross-roads and red and green circular disks drawn on an underlying sheet, so that by a slight turn red and green could be interchanged on the whole plan.

13 February 1977: Bastiaan swings the gate of a labor yard. The door drew traces in the sand. "What is it?" I ask him. "A line," he says. "What kind?" A curved line." "What kind of curved line?" "A circle." "Yes," I said, "about a quarter of a circle. It is like the hands of a clock." "No," he said, "like wipers."

21 February 1977: Bastiaan played with a little car in the corridor of my office. As usual, he made a written report--awfully spelled. "9 times pushing a car comes through the corridor of IOWO." He remarks that with another car it would be different, for instance, if the car were longer or the wheels stiffer.

18 March 1977: From the bar on the 21st floor of the Holiday Inn, we were looking down on the Railroad Station. He saw "sparks of a train." I explained to him that the sparks were really reflections of the sun in the windows of the moving train. I made a drawing and asked him how a sun ray would be reflected by a window. When he falsely produced the ray, I made a new drawing. His answer was again wrong, but then he corrected himself. I asked similar questions about a circular billard table. His constructions were reasonable.

Freudenthal

24 March 1977: At dinner, with paper napkins (folded twice as usual), I unfolded a napkin, asking Bastiaan whether by folding he could make a square half the size. He folded it back and immediately saw he was wrong. He tried it by folding away small strips parallel with the borders. "It should be done more precisely," I explained to him, "you must fold it differently." He seized a corner in order to fold it towards the center. Unfortunately, I intervened to prevent him from going beyond the center. So he stopped at the center and immediately applied the procedure to the other corners. In the same way, he then halved the new square.

14 April 1977: Bastiaan tried to describe the size of a certain box. He did it with his hands, "that wide, that high." He did not understand my question, "How long?" Cautiously, I tried to lead him to talk about the length. "I cannot understand what you mean, Grandpa." Using a pillar box, I pointed out that three dimensions are required. He then understood my question, though he described the third dimension also with his pair of hands lifted left and right, rather than in front. Later, when describing a parked car it appeared that he wholly understood the matter.

16 April 1977: Bastiaan was playing on an exercise trail. The trail was 80 m long. I followed him walking. He ran back to meet me and then ran to the finish. "I have run twice 80 meters." "Why, you did not return to the start." "I went to the middle and back, and 40 + 40 = 80." At home he wanted to tell about how he ran 2 x 8 meters, but he makes it 600. He did not see the connection with 8 + 8 = 16, which he knew very well.

August 1977: In our holiday resort, at the edge of a brook, Bastiaan makes a model of the North Sea, the Dutch coast, the Frisian Islands, and the German and Danish coast. He called it a miniature (possibly he learned the word in connection with golf). His grandmother took a picture of his work. He said: "This becomes a double miniature."

1 September 1977: During dinner at Bastiaan's home, he asks how much goes in a wine bottle. I told him a liter. He asked, "What is a liter?" I explained that a liter of water weighs 1 kg. He objected: "But a liter of something else weighs more." I continued, "According to a 750 at the bottom, it should be 3/4 liter." Bastiaan asked, "What is three quarters?"

I showed Bastiaan with a distance of 1 to 2 dm: "This is half, this is half of a half, a quarter, and that is three quarters." He again protested: "If you start this way," his arms extended, "it is much more." There are two pieces of beefsteak left on a plate--one somewhat bigger than the other. Miss Adda pointed to the bigger piece and asked: "Is this half of it?" Bastiaan said, "Half of what?" And immediately continued: "No, it is bigger."

Later in the meal, Bastiaan said spontaneously, "Parallel lines do not meet." He probably had learned this fact at school. I asked him whether lines that do not meet are always parallel. After some experimenting with two forks he exclaims: "Two roads above each other, they never meet, they go far from each other."

10 September 1977: Back from a long walk, we cross a slightly ascending bridge--Monica and myself on the side walk, Bastiaan on a small wall along the sidewall ascending discontinuously by steps with horizontal pieces about 5 meters wide. Bastiaan said: "Now you are higher, but then I will be higher." He was referring to the difference between continuous and discontinuous ascent (see Figure 8).

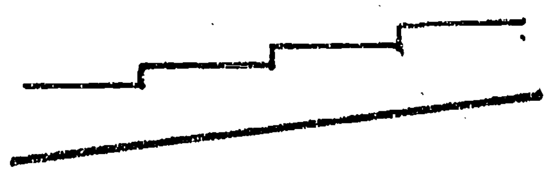

Fig... . Continuous Versus Discontinuous Ascent

26 October 1 '7: On a walk to church, Bastiaan wanted to know the height of a rather tall tower. I suggested that he should estimate it. He guessed, "100 meters." I criticized the estimation. "You know the cathedral tower is only a bit taller than 100 meters. I am 1.80 m tall and <u>that</u> is nearly 2 meters." Then I stood by the tower and asked, "How much then is the whole tower?" Bastiaan responded angrily: "I did ask <u>you</u> to tell me how high the tower is."

I went with him to the low stone wall surrounding the broad forecourt of the church. "Give me the stick," I said. (It was a piece of wood of about 40 cm with a sharp tip which he had found just before.) I put the stick vertically upon the wall, pressed my right cheek upon the wall such that my right eye, the tip of the stick, and the spire of the tower were in one line. I asked him to do as I did. He understood my intention as he looked with the correct eye. He felt something should be measured though he was not sure what it might be. He measured the distance between the place of his eye and the stick with a span between thumb and little finger* (which he knew was 1 dm). The distance was 3 dm. "So the tower is 3 meters," he said. "That is impossible," I answered. I suggested for him to measure the distance between the wall and the tower. While he preferred to estimate it, I insisted on the distance being measured by steps. It was 50 steps. "So 50 meters." I told him his steps were not a meter, as I had done many times before. "Even my steps are only 80 cm." Then, I paced the distance and told him it was 36 m. He repeated the observation on the wall and spontaneouly remarked that the stick was somewhat longer than the distance from eye to stick. "So, <u>how</u> tall will the tower be?" He grasps that it must be higher than 35 m. I suggested, "40 meters."

* Compare this method with that of 27 October 1976.

Freudenthal

I continued, "And how high is the clock?" He again starts aiming, and marks the height of the clock on the stick. With some help this is established as 3/4 of the stick. Then, without any hints, he estimated the height of the clock to be 30 meters.

Meanwhile, according to his habit, he made a written report. He drew the sketch (see Figure 9) without help; and, I did not correct it. It shows that he understood the essentials.

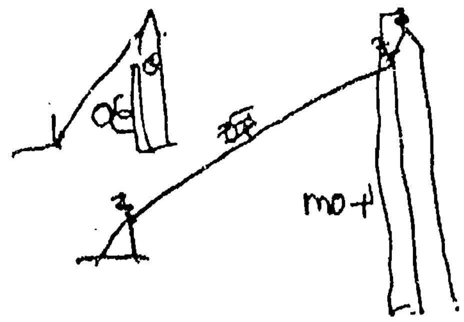

Figure 9. Bastiaan's Drawing of the Church Tower

Next, Bastiaan remarked: "It is a better way to do it with a little mirror. It is a pity I did not pick up the mirror pieces at the locks." Indeed, on our way back we found pieces of a smashed car mirror. Bastiaan explained in exact terms how he could measure the height of a tower using a mirror. His alternative solution is shown in the next figure.

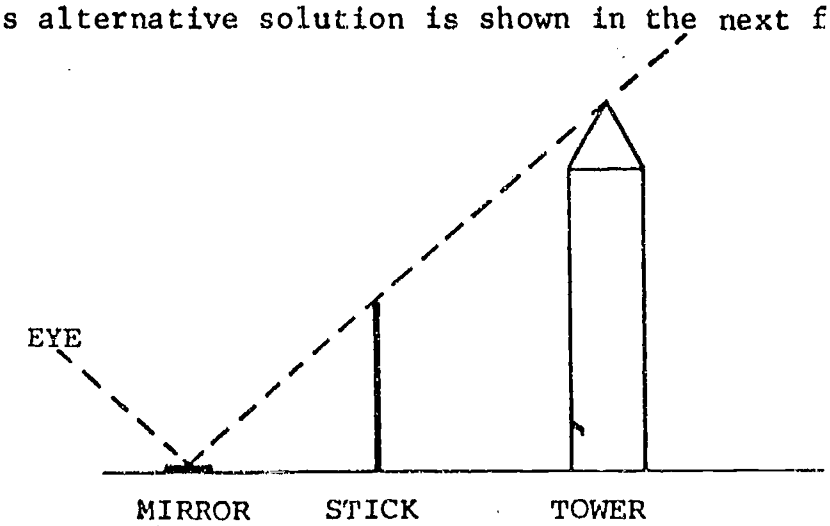

Figure 10. Bastiaan's Alternative Solution to the Tower Problem

"It is easier," he said, "You need not press your eye upon the wall, but it also is more difficult because it is more difficult to find." I tried to explain to him that with sunshine the height can also be found using shadows. "No, you cannot get the whole shadow on the court," he objected.

At the locks he picked up a mirror piece and tried his technique on the height of streetlamps--unsuccessfully.

3 December 1977: Looking at a brush, Bastiaan said: "I ere they forgot 1 cm^3 to be trimmed." (It was meant as a joke.) I asked him what

a cm^3 is. He replied, "A cube with all sides 1 cm." "It is called an edge rather than a side. How many edges does a cube have?" Bastiaan answered (hesitating): "8." I said: "4 below, 4 above, and..." (he seemed to figure out 6 x 4.) I continued asking: "How many corners does a cube have?" "8." "How many faces?" "6." "How many edges? 4 below, 4 above, and ... standing ones?" Bastiaan answered, "12." "A little while ago, you tried to do 6 x 4, didn't you? What is wrong with that?" Bastiaan said, "You did not say standing ones."

Summary and Conclusions

Let us stop to think about the meaning of the preceding bulk of observations. It is rough material that I indifferently collected and have used extensively in the meantime to understand geometrical learning.

My method widely differs from Piaget's I did not start from any preconceived theory, and I rejected the idea that geometrical development proceeds according to some logical system of geometry however beautiful it might be. So I could not confirm that geometry develops according to the classification of the Erlanger Programme. I should add that even as a mere hypothesis this is highly improbable in itself. As a matter of fact, Piaget's experiments (which I do not consider as meaningful), if correctly interpreted, contain as many proofs to the contrary There is as little evidence that geometry develops according to the sequence "topology, projective, affine, euclidean geometry" as there is for plane and space to be mentally constituted as cartesian products of two or three straight lines, as Piaget puts it.

I would not maintain that I can do without some theory of my own, albeit as a frame to order and to understand my observations. I believe that little children have at their disposal certain mental objects--in particular, geometric ones--and mental operations on these objects; and my efforts are aimed at discovering them. It would be a hard thing to decide which are innate, and to what degree, and which are acquired at a very early age. Straight lines, parallelism, circles, squares, right angles, planes, symmetries, congruences and similarities are suggested so early and by so many concrete objects and phenomena of our cultural environment that there is little chance to trace back their origin to an even earlier source as in innate ideas. On the other hand, we cannot but assume that on our cerebral cortex a computer programme is imprinted that allows us to perform congruence and similarity transformations and use polar coordinates to compare things as to their "true size" and to put things "upright."

Geometry is a part of mathematics where one can go a long way with nothing but mental objects. Many people never form concepts like straight line, circle, square, and other ideas mentioned in this paper. Without formal education, those who go as far as to form concepts remain a small minority.

Formal instruction of geometry usually aims at teaching concepts. I think this is wrong. Teaching geometry should start with developing mental objects, and this development should begin at the earliest age where it is feasible--that is, during the whole period of kindergarten and elementary school. Starting geometry as late as is now the habit is a serious mistake.

Let us once more review some of the stories from Bastiaan's diary. With young children one example may suffice to explain what a circle is, or a square, or a horse, or a tree. Probably somewhat more examples, though not very many, are needed to tell them about "two" or "three." Color seems to need even more examples. But what about grasping the precision required (or the vagueness allowed) if something is to be identified as a circle? How did Bastiaan grasp that in the world of trees the cross-sections deserve the predicate of circle while sharp holes in metal sheets have to fulfill much stronger conditions to be admitted as circles?

Picture books and playthings which imitate objects in the world of adults presuppose and develop ideas about ratio and proportion long before they are verbalized and conceptualized. But what about constructing ratios in order to grasp big sizes? I often constructed such ratios to explain about sizes in the universe. Bastiaan used these ratios to estimate the size of clouds.

Straight lines are very early mental objects. Straightness is suggested by objects and phenomena in the normal environment: by the upright posture, by the extended limbs--hands, legs, fingers--by the stalks of plants and the trunks of trees, and by the straight path (which is also the shortest, the most direct path).

One of the first tools made by man is the arrow, paragon of straightness, and civilization produces ever and ever more and more objects and processes and elicits actions suggesting or representing straightness: sticks, pins, rims, edges, paths, folds, cuts, tended strings.

Straight lines originate in a multifarous way:

by copying (drawing with the ruler),
as intersection of planes,
as a cut line,
as a fold line,
as a straight-on path,
as a shortest path,
as a stretched string,
as a vision line (light ray),
as a reflection axis (in the plane),
as a rotation axis (in space).

These ways of generating straight lines are not independent of each other. The sharp edge of a ruler is someth.ng like the intersection of two planes. The cut line originates as it were by copying the sharp edges of the pairs of scissors. The fold line comes into being by giving the paper the shape of two (fragments of) planes, and the straightness of the axis of reflection may somehow be explained similarly.

The most subtle in this catalogue seems to be the vision line. What was most astonishing about Bastiaan's measuring the church tower--and his most original idea--was connecting his eye, the tip of the stick, and the spire of the tower by a straight line. I would not have expected it so early. It is strange that explaining shadows geometrically is a later stage.

The catalogue of sources of the straight line is intended to show the phenomeno-logical complexity of such a mental object. Is it developmentally as complex? There are innumerable questions I could ask about this one mental object and among them there are very few I would be able to discuss. What is general about learning processes is problems rather than solutions.

Reference Note

1. Pieces from the diary were published in Dutch: Wandelingen met Bastiaan, Bastiaan's Lab, Bastiaan meet (de wereld); Pedomorfose 7 (1975), no. 25, p. 51-64; Pedomorfose 8 (1976), no. 30, p. 35-54; Pedomorfose 10 (1978), no. 37, p. 62-68.

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